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A New Treatment of Optical Aberrations

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V. *A New Treatment of Optical Aberrations.*

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The method developed by Gauss in his 'Dioptrische Untersuchungen' is probably the most powerful, as well as the readiest, method in geometrical optics. It has in effect hitherto been restricted to systems in which the relations of original and emergent rays are strictly linear, or, in optical language, those in which the aberrations can be neglected. It is true that Seidel bases his celebrated discussion of aberrations upon Gauss's method, but he soon modifies it and replaces its system of co-ordinates and characteristic steps by others. The following pages show how the method may be extended and retained throughout the discussion of the aberrations of any co-axial system. They will be found to throw light upon the general relationships of the well-known Petzval condition and Abbe Sine condition, to furnish a ready method of describing, analysing and measuring the faults of an optical image, and to be particularly adapted to numerical calculations, to the order to which these are necessary for telescopic objectives.

It will be convenient to state here the essentials of the method in the form in which they will be used later. Let Oxyz, O$'x'y'z'$ be rectangular axes in the original and emergent media, of which the refractive indices are μ, μ' respectively. Ox, O$'x'$ are the axes of the optical system. Take the equations of any ray before and after its passage through the system in the respective forms

$$y = \beta x+b, \qquad z = \gamma x+c,$$

and

$$y' = \beta' x'+b', \qquad z' = \gamma' x'+c', \qquad \ldots\ldots\ldots (1)$$

then, provided there is a strict linear correspondence as well as symmetry about the axis, we may put

$$b' = gb+h\beta, \qquad c' = gc+h\gamma,$$

$$\beta' = kb+l\beta, \qquad \gamma' = kc+l\gamma, \qquad \ldots\ldots\ldots (2)$$

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where g, h, k, l are constants involving the curvatures of the refracting surfaces, the distances between them and the refractive indices; also

$$gl-hk = \mu/\mu'.$$

Following SEIDEL, we shall call such systems normal systems.

In particular, for a single refracting surface,

$$2x = \mathrm{B}(y^2+z^2)+\ldots,$$

without change of origin, the scheme

$$\begin{Bmatrix} g, & h \\ k, & l \end{Bmatrix},$$

as I shall call it, becomes

$$\begin{Bmatrix} 1, & * \\ -\left(1-\frac{\mu}{\mu'}\right)\mathrm{B}, & \frac{\mu}{\mu'} \end{Bmatrix}$$

where * is put in place of zero. Or again, a simple shift of origin by a distance d may be represented by the scheme

$$\begin{Bmatrix} 1, & d \\ *, & 1 \end{Bmatrix}.$$

If two instruments be represented by the schemes

$$\begin{Bmatrix} g_1, & h_1 \\ k_1, & l_1 \end{Bmatrix}, \qquad \begin{Bmatrix} g_2, & h_2 \\ k_2, & l_2 \end{Bmatrix},$$

light passing through (1) first and then through (2), and the emergent origin for the first being made the same as the original origin of the second, their combined effect is given by the scheme

$$\begin{Bmatrix} g_1g_2+k_1h_2, & h_1g_2+l_1h_2 \\ g_1k_2+k_1l_2, & h_1k_2+l_1l_2 \end{Bmatrix}, \quad \ldots \ldots \quad (3)$$

which may be written down by multiplying the rows of the later scheme into the columns of the former, as if they were determinants. It will be shown hereafter that this rule is remarkably well adapted for numerical calculation—a fact that does not seem to have been remarked before. The scheme corresponding to any system, as, for example, any thick lenses, arranged at intervals along an axis, may be built up from its elements by this rule, by writing down the schemes in order belonging to the successive refracting surfaces and shifts of origin, and compounding these; if we have to compound in this manner a sequence of schemes

$$\{g_1, \ldots\}, \quad \{g_2, \ldots\}, \quad \{g_3, \ldots\}, \quad \ldots, \quad \{g_n, \ldots\},$$

then, provided we do not change the order in which the schemes present themselves, the composition may be effected in such groups as may be convenient, and may be performed either from left to right or from right to left.*

The product of the determinants of the component scheme $(g_1l_1-h_1k_1)$, $(g_2l_2-h_2k_2)$, ... gives the value of the determinant $GL-HK$ of the compounded scheme.

The analytical scheme corresponding to any instrument of which the cardinal points are known may be written down at sight, and conversely, by the relations

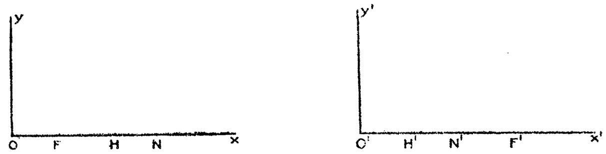

Fig. 1.

$$OF = l/k, \quad HF = n/k, \quad NF = 1/k; \qquad O'F' = -g/k, \quad H'F' = -1/k, \quad N'F' = -n/k,$$

where

$$n = gl-hk \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and H, H′, N, N′, F, F′ denote, as usual, the unit points, nodal points, and principal foci respectively.

If it is desired to work geometrically, we may set the original and emergent axes across one another in a figure at any angle, H and H′ being superposed (or else

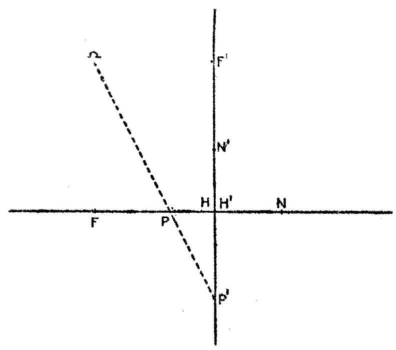

Fig. 2.

N, N′), and finding a point Ω with co-ordinates HF, H′F′; then any straight line ΩPP′ through Ω determines points PP′, which are conjugate foci.

The following method of compounding any two given systems may also be mentioned :—

* A general discussion of the linear system by the author will be found in 'Proc. London Math. Soc.,' vol. 29, p. 33.

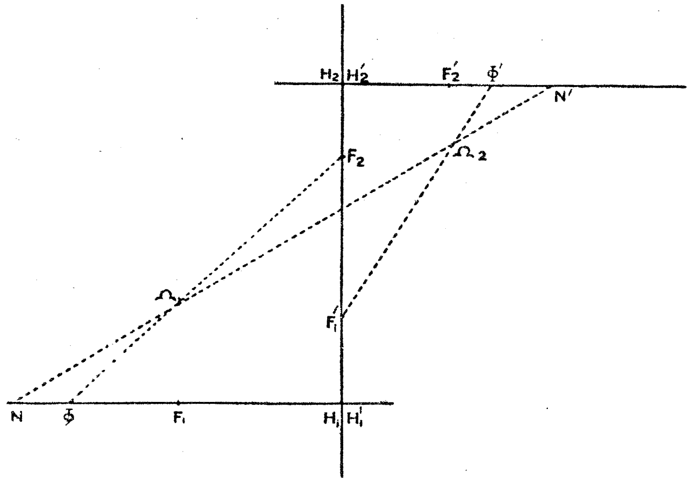

Fig. 3.

Set the axes as shown in the figure, the distance between the points $H_1H'_1$ and $H_2H'_2$ being annulled. Then the lines $F_2\Omega_1$, $F'_1\Omega_2$ give Φ, Φ', the principal foci, and the line $\Omega_1\Omega_2$ gives points N, N′ which are conjugate to one another and are the nodal points of the compound system.

We see that it is always possible to determine a geometrical system that shall correspond to any given values of g, h, k, l. Thus, for example, $n = 0$ implies that F′ is conjugate to *every* point of the original system, or, what is the same thing, that every emergent ray goes through F′.

If the emergent origin is at the principal focus, $g = 0$.

If the original origin is at the principal focus, $l = 0$.

If the original and emergent origins are conjugate points, $h = 0$.

We shall now consider the case of refraction of a general ray at a symmetrical surface centred upon the x-axis and shall show that a scheme $\{g+\delta g, \ldots\}$ may be derived for it, which shall include the aberrations; these, represented by the additional terms $\delta g, \ldots$, will, of course, vary from point to point with the squares and products of the co-ordinates and angles of incidence upon the surface, whereas for the pure linear scheme $g, \ldots$ are the same for every ray of the beam.

Taking rectangular axes $Oxyz$, let the equation of the surface separating the region of index μ from that of index μ' be

$$2x = B(y^2+z^2)+\tfrac{1}{4}C(y^2+z^2)^2. \qquad (5)$$

Let a ray

$$y = \beta x+b, \qquad z = \gamma x+c,$$

in the original medium be transformed by refraction at the surface into

$$y' = \beta' x'+b', \qquad z' = \gamma' x'+c',$$

where the axes are, in fact, the same but are accented to indicate the difference of medium.

The positive direction of the x-axis is that in which the light is travelling.

In the diagram P_0 is the point where the original and emergent rays meet at the surface, PP_0 is the original ray, $P'P_0$ the emergent ray, and $(0, b, c)$ are the co-ordinates of P, $(0, b', c')$ those of P′, and we shall take (a_0, b_0, c_0) as those of P_0.

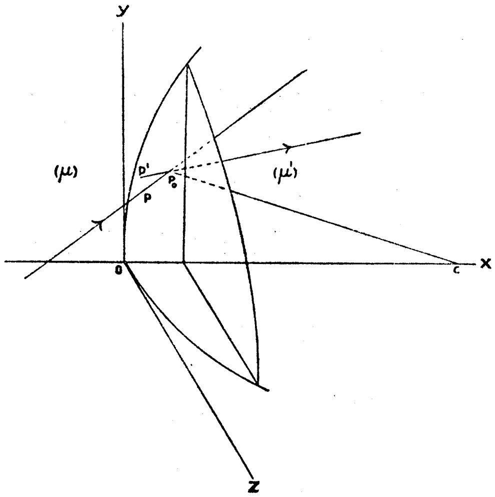

Fig. 4.

If (l, m, n) (l', m', n') are the direction cosines of an original and emergent ray, (p, q, r) those of the normal to the surface at the point of incidence, we have the known equations

$$(\mu l-\mu' l')/p = (\mu m-\mu' m')/q = (\mu n-\mu' n')/r = \mu \cos\theta-\mu'\cos\theta',$$

where θ, θ' are the angles made by the two rays and the normal.

Now

$$l = m/\beta \qquad = n/\gamma \qquad = 1-\tfrac{1}{2}\beta^2-\tfrac{1}{2}\gamma^2$$
$$l' = m'/\beta' \qquad = n'/\gamma' \qquad = 1-\tfrac{1}{2}\beta'^2-\tfrac{1}{2}\gamma'^2$$
$$-p = q/\mathrm{B}b_0+\tfrac{1}{2}\mathrm{C}b_0(b_0^{\,2}+c_0^{\,2}) = r/\mathrm{B}c_0+\tfrac{1}{2}\mathrm{C}c_0(b_0^{\,2}+c_0^{\,2}) = -1+\tfrac{1}{2}\mathrm{B}^2b_0^{\,2}+\tfrac{1}{2}\mathrm{B}^2c_0^{\,2} = -1+\tfrac{1}{2}q^2+\tfrac{1}{2}r^2$$

if we neglect higher powers of the small quantities.

Further

$$\cos\theta = 1-\tfrac{1}{2}\theta^2, \qquad \cos\theta' = 1-\tfrac{1}{2}\theta'^2,$$

where

$$\theta^2 = (\beta-q)^2+(\gamma-r)^2, \qquad \theta'^2 = (\beta'-q)^2+(\gamma'-r)^2,$$

and we have approximately

$$\mu(\beta-q) = \mu'(\beta'-q), \qquad \mu(\gamma-r) = \mu'(\gamma'-r).$$

Substituting above for m, m', we have

$$\mu\beta(1-\tfrac{1}{2}\beta^2-\tfrac{1}{2}\gamma^2)-\mu'\beta'(1-\tfrac{1}{2}\beta'^2-\tfrac{1}{2}\gamma'^2) = (\mu-\mu')\,q-\tfrac{1}{2}q\,(\mu\theta^2-\mu'\theta'^2).$$

But since

$$\theta/\mu' = \theta'/\mu,$$

therefore

$$\mu\theta^2 - \mu'\theta'^2 = -(\mu-\mu')\,\theta\theta',$$

and

$$\theta\theta' = (\beta-q)(\beta'-q)+(\gamma-r)(\gamma'-r).$$

Hence the right-hand member of this equation reads

$$\begin{aligned}(\mu-\mu')\,q\,[1+\tfrac{1}{2}(\beta\beta'+\gamma\gamma')+\tfrac{1}{2}(q^2+r^2)-\tfrac{1}{2}\beta q-\tfrac{1}{2}\beta' q-\tfrac{1}{2}\gamma r-\tfrac{1}{2}\gamma' r]\\ = q\,[(\mu-\mu')+\tfrac{1}{2}(\mu-\mu')(q^2+r^2)\\ +\tfrac{1}{2}(\mu-\mu')(\beta\beta'+\gamma\gamma')-\tfrac{1}{2}(\beta+\beta')(\mu\beta-\mu'\beta')-\tfrac{1}{2}(\gamma+\gamma')(\mu\gamma-\mu'\gamma')]\\ = q\,[(\mu-\mu')+\tfrac{1}{2}(\mu-\mu')(q^2+r^2)-\tfrac{1}{2}\mu(\beta^2+\gamma^2)+\tfrac{1}{2}\mu'(\beta'^2+\gamma'^2)],\end{aligned}$$

or, since

$$q = -\mathrm{B}b_0(1-\tfrac{1}{2}q^2-\tfrac{1}{2}r^2)-\tfrac{1}{2}\mathrm{C}b_0(b_0{}^2+c_0{}^2)$$

and

$$b_0 = b+a_0\beta = b+\tfrac{1}{2}\beta(q^2+r^2)/\mathrm{B},$$

therefore

$$q(1+\tfrac{1}{2}q^2+\tfrac{1}{2}r^2) = -b\,[\mathrm{B}+\tfrac{1}{2}\mathrm{C}(b^2+c^2)]-\tfrac{1}{2}\beta(q^2+r^2),$$

and the equation becomes

$$\begin{aligned}\beta\,[\mu-\tfrac{1}{2}\mu(\beta^2+\gamma^2)+\tfrac{1}{2}(\mu-\mu')(q^2+r^2)]-\beta'\,[\mu'-\tfrac{1}{2}\mu'(\beta'^2+\gamma'^2)]\\ = -b\,[(\mu-\mu')\,\mathrm{B}+\tfrac{1}{2}(\mu-\mu')\,\mathrm{C}(b^2+c^2)+\tfrac{1}{2}\mu\mathrm{B}(\beta^2+\gamma^2)+\tfrac{1}{2}\mu'\mathrm{B}(\beta'^2+\gamma'^2)],\end{aligned}$$

or dividing by the coefficients of β' and writing

$$\mu/\mu' = n,$$

$$\begin{aligned}\beta' = b\,[-(1-n)\,\mathrm{B}-\tfrac{1}{2}(1-n)\,\mathrm{C}(b^2+c^2)+\tfrac{1}{2}n\mathrm{B}(\beta'^2+\gamma'^2-\beta^2-\gamma^2)]\\ +\beta\,[n+\tfrac{1}{2}n(\beta'^2+\gamma'^2-\beta^2-\gamma^2)-\tfrac{1}{2}(1-n)(q^2+r^2)].\end{aligned}$$

Also

$$b_0 = b+a_0\beta = b'+a_0\beta'.$$

Therefore

$$b' = b+a_0(\beta-\beta');$$

but approximately

$$\beta' = -(1-n)\,\mathrm{B}b+n\beta;$$

therefore

$$b' = b\,[1+\tfrac{1}{2}(1-n)(q^2+r^2)]+\beta\,[\tfrac{1}{2}(1-n)(q^2+r^2)/\mathrm{B}];$$

or if we write

$$\left.\begin{aligned}\omega &= \tfrac{1}{2}(1-n)(q^2+r^2) = \tfrac{1}{2}(1-n)\,\mathrm{B}^2(b^2+c^2),\\ \psi &= \tfrac{1}{2}n(\beta'^2+\gamma'^2-\beta^2-\gamma^2),\end{aligned}\right\}\quad . \ . \ . \ . \ . \ . \ (6)$$

we may put

$$\left.\begin{aligned}b' &= b\,[1+\omega]+\beta\,[\omega/\mathrm{B}],\\ \beta' &= b\left[-(1-n)\,\mathrm{B}+\mathrm{B}\psi-\frac{\mathrm{C}}{\mathrm{B}^2}\,\omega\right]+\beta\,[n+\psi-\omega].\end{aligned}\right\}\quad . \ . \ . \ . \ . \ . \ (7)$$

In the same way it follows

$$c' = c[1+\omega] + \gamma[\omega/B],$$
$$\gamma' = c\left[-(1-n)B + B\psi - \frac{C}{B^2}\omega\right] + \gamma[n+\psi-\omega]; \quad \ldots \ldots \quad (8)$$

for the case of the paraboloid

$$C/B^2 = 0,$$

sphere

$$C/B^2 = B.$$

We shall generally write

$$C/B^2 = \epsilon B.$$

We remark that the coefficients that transform the (b, β) system into (b', β') are the same as those which transform (c, γ) into (c', γ'), and for any surface each is expressed in terms of the two functions ψ, ω defined by equation (6), in addition to the refractive index and curvature.

These equations therefore permit us to treat rays which cross the axis with the same readiness as those which intersect it, a thing which is very troublesome in the trigonometrical discussion of the question. They also apply equally easily to the sphere, the paraboloid, and any intermediate form.

Before proceeding with the discussion of these formulæ I shall verify that they cover the known expression for longitudinal aberration on the axis after refraction at a single spherical surface, as it is given in the text books.

Suppose the ray meets the axis at $x = v$, $x' = v'$, so that

$$b + \beta v = 0, \qquad b' + \beta' v' = 0;$$

then the equation connecting v, v' for the case of the sphere is

$$-vv'[-(1-n)B + B(\psi-\omega)] + v'[n+\psi-\omega] - v[1+\omega] + \omega/B = 0,$$

or, dividing by vv' and rearranging the terms,

$$\frac{1}{v'} - \frac{n}{v} - (1-n)B = \left(B - \frac{1}{v}\right)\left(-\psi + \omega - \frac{\omega}{Bv'}\right);$$

but

$$\psi = \tfrac{1}{2}n(\beta'^2 - \beta^2) = \tfrac{1}{2}nb^2\left(\frac{1}{v'^2} - \frac{1}{v^2}\right) = \tfrac{1}{2}nb^2\left(\frac{1}{v'} - \frac{1}{v}\right)\left(\frac{1}{v'} + \frac{1}{v}\right)$$

and

$$\frac{B - \frac{1}{v'}}{n} = \frac{B - \frac{1}{v}}{1} = \frac{\frac{1}{v'} - \frac{1}{v}}{1-n};$$

also

$$\frac{\omega}{B} = \tfrac{1}{2}(1-n)Bb^2 = \tfrac{1}{2}b^2\left(\frac{1}{v'} - \frac{n}{v}\right).$$

Hence the right-hand member above is equal to

$$\left(\mathrm{B}-\frac{1}{v}\right)\left[-\tfrac{1}{2}b^2(1-n)\left(\mathrm{B}-\frac{1}{v'}\right)\left(\frac{1}{v'}+\frac{1}{v}\right)+\tfrac{1}{2}b^2\left(\mathrm{B}-\frac{1}{v'}\right)\left(\frac{1}{v'}-\frac{n}{v}\right)\right]$$

$$=\tfrac{1}{2}b^2\left(\mathrm{B}-\frac{1}{v}\right)\left(\mathrm{B}-\frac{1}{v'}\right)\left[-(1-n)\left(\frac{1}{v'}-\frac{1}{v}\right)+\left(\frac{1}{v'}-\frac{n}{v}\right)\right]$$

$$=\tfrac{1}{2}b^2\left(\mathrm{B}-\frac{1}{v}\right)\left(\mathrm{B}-\frac{1}{v'}\right)\left(\frac{n}{v'}-\frac{1}{v}\right).$$

This is one of the usual expressions; compare HERMAN'S 'Optics,' p. 189, (iii.). After a slight transformation it leads to the Zinken-Sommer expression for the separation of the focal lines in any co-axial system, and thence, as WHITTAKER has shown ('Theory of Optical Instruments,' p. 26), to the expressions of SEIDEL'S theory.

We may verify also that these expressions lead to the known results in the case of the parabolic mirror.

Consider the focus for rays parallel to the axis, *i.e.*, when $\beta = \gamma = 0$.

$$0 = 1+\omega+v'[-(1-n)\,\mathrm{B}+\mathrm{B}\psi].$$

But

$$n = -1, \qquad \omega = \tfrac{1}{2}(1-n)(q^2+r^2) = \mathrm{B}^2(b^2+c^2),$$

$$\psi = \tfrac{1}{2}n(\beta'^2+\gamma'^2) = \tfrac{1}{2}n(1-n)^2\,\mathrm{B}^2(b^2+c^2) = -2\mathrm{B}^2(b^2+c^2),$$

so that

$$v' = [1+\mathrm{B}^2(b^2+c^2)]/[2\mathrm{B}+2\mathrm{B}^3(b^2+c^2)] = 1/2\mathrm{B},$$

so that the longitudinal aberration vanishes at the principal focus. More generally, the ray $y' = \beta'x+b'$, $z' = \gamma'x+c'$, corresponding to a general incident ray for which, say, $\gamma = 0$, meets the plane $x' = d'$ in points whose co-ordinates are

$$y' = b[-(1-n)\,\mathrm{B}d'+1+\mathrm{B}d'\psi+\omega]+\beta[d'(n+\psi-\omega)+\omega/\mathrm{B}],$$
$$z' = c[-(1-n)\,\mathrm{B}d'+1+\mathrm{B}d'\psi+\omega].$$

For the paraboloidal reflector

$$\omega = \mathrm{B}^2(b^2+c^2),$$
$$\psi = \tfrac{1}{2}n(\beta'^2+\gamma'^2-\beta^2) = -2\mathrm{B}^2(b^2+c^2)-2\mathrm{B}b\beta,$$

since $\beta' = -\beta-2\mathrm{B}b$, $\gamma' = 2\mathrm{B}c$; therefore taking the focal plane $d' = 1/2\mathrm{B}$, we have

$$y' = -\beta/2\mathrm{B}-\mathrm{B}b^2\beta-\beta[\tfrac{1}{2}\mathrm{B}(b^2+c^2)+b\beta],$$
$$z' = -\mathrm{B}bc\beta\,; \qquad \qquad (9)$$

these are known expressions, leading to the theory of the coma of a parabolic reflector; *cf.* PLUMMER, 'Mon. Not. R. A. S.,' LXII., p. 365, (9), (10).

If we compare the scheme

$$\begin{Bmatrix} 1+\omega, & \omega/\mathrm{B} \\ -(1-n)\,\mathrm{B}+\mathrm{B}\psi-\epsilon\mathrm{B}\omega, & n+\psi-\omega \end{Bmatrix} \quad \ldots \ldots \quad (7), (8)$$

with the equations (4), we see the aberrational scheme is the equivalent of a linear scheme, for a single surface, for which the plane of origin passes through the point $x = \omega/(1-n)\,\mathrm{B}$, that is, through the actual point of incidence $x = \frac{1}{2}\mathrm{B}\,(b_0^2+c_0^2)\ldots$; if we take the curvature as $\mathrm{B}\,(1+\frac{1}{2}\epsilon q^2+\frac{1}{2}\epsilon r^2)$, and the refractive indices $\mu\,(1-\frac{1}{2}\beta^2-\frac{1}{2}\gamma^2)$, $\mu'\,(1-\frac{1}{2}\beta'^2-\frac{1}{2}\gamma'^2)$ respectively.

As to the latter it may be noticed that the exact equation for refraction of a ray impinging at the origin, and in the plane Oxy, is

$$\mu \sin(\tan^{-1}\beta) = \mu' \sin(\tan^{-1}\beta')$$

or, to our order,

$$\mu\,(1-\tfrac{1}{2}\beta^2)\,.\,\beta = \mu'\,(1-\tfrac{1}{2}\beta'^2)\,.\,\beta'.$$

Hence the aberration ψ may be described as due purely to the obliquity of the ray to the axis, and the aberration ω to the lateral separation from the axis, and we see that the somewhat remarkable fact that two functions ω, ψ suffice to express the aberrations of every ray may be stated in the form that there is no term which is produced jointly by obliquity and lateral separation.

If in any instrument we have a number of surfaces each introducing aberrational terms, and if the schemes preceding and following the surface (r) be compounded so as to read, say, $\{g, \ldots\}$, $\{g', \ldots\}$, then the whole may be represented by

$$\begin{Bmatrix} g, & h \\ k, & l \end{Bmatrix} \begin{Bmatrix} 1+\omega_r, & \omega_r/\mathrm{B}_r \\ -(1-n_r)\,\mathrm{B}_r+\mathrm{B}_r\psi_r-\epsilon_r\mathrm{B}_r\omega_r, & n_r+\psi_r-\omega_r \end{Bmatrix} \begin{Bmatrix} g', & h' \\ k', & l' \end{Bmatrix} \quad . \quad . \quad (10)$$

and the portions added to the general scheme in consequence of the aberrations of the r^{th} surface will be

$$\begin{Bmatrix} \delta\mathrm{G}_r, & \delta\mathrm{H}_r \\ \delta\mathrm{K}_r, & \delta\mathrm{L}_r \end{Bmatrix} = \omega_r \begin{Bmatrix} g, & h \\ k, & l \end{Bmatrix} \begin{Bmatrix} 1 & 1/\mathrm{B}_r \\ -\epsilon_r\mathrm{B}_r, & -1 \end{Bmatrix} \begin{Bmatrix} g', & h' \\ k', & l' \end{Bmatrix} + \psi_r \begin{Bmatrix} g, & h \\ k, & l \end{Bmatrix} \begin{Bmatrix} * & * \\ \mathrm{B}_r, & 1 \end{Bmatrix} \begin{Bmatrix} g', & h' \\ k', & l' \end{Bmatrix} (11)$$

and in this the schemes $\{g, \ldots\}$, $\{g', \ldots\}$ may be taken at their "normal" values without regard to aberrations introduced by surfaces other than the surface (r). If then we write, adding the effects of all the surfaces,

$$\delta\mathrm{G} = \Sigma_r \delta\mathrm{G}_r, \quad \ldots \ldots \ldots \ldots \quad (12)$$

and if we now denote by $\{g, \ldots\}$ the scheme got by compounding all the normal schemes of the instruments in succession, whether these are refractions, or mere shifts of origin from one surface to the next, we have for the relation between any original ray

$$y = \beta x + b, \qquad z = \gamma x + c,$$

and the corresponding emergent ray

$$y' = \beta' x' + b', \qquad z' = \gamma' x' + c',$$

$$\left.\begin{aligned}(b, \beta)\begin{Bmatrix}G+\delta G, & H+\delta H\\ K+\delta K, & L+\delta L\end{Bmatrix} &= (b'+\delta b', \beta'+\delta\beta'),\\ (c, \gamma)\begin{Bmatrix}G+\delta G, & H+\delta H\\ K+\delta K, & L+\delta L\end{Bmatrix} &= (c'+\delta c', \gamma'+\delta\gamma'),\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (13)$$

where b', β', c', γ' are the values that would result if δG, ... were all zero; and turning back to the expressions (6) we see that ω_r, ψ_r are expressed in terms of the incidence—(b, β, c, γ)—upon the plane Oyz by the equations

$$\omega_r = \tfrac{1}{2}(1-n_r)\, B_r^2 (b_r^2 + c_r^2),$$

where

$$b_r = g_r b + h_r \beta, \qquad c_r = g_r c + h_r \gamma;$$

$$\psi_r = \tfrac{1}{2} n_r (\beta'^2_r + \gamma'^2_r - \beta_r^2 - \gamma_r^2), \quad . \; . \; . \; . \; . \; . \quad (14)$$

where

$$\beta_r = k_r b + l_r \beta, \qquad \gamma_r = k_r c + l_r \gamma,$$

$$\beta'_r = k_{r+1} b + l_{r+1} \beta, \qquad \gamma'_r = k_{r+1} c + l_{r+1} \gamma.$$

We notice that the determinants—$(gl-hk)$—of the scheme multiplied by each ψ_r will always be zero, and that of the scheme multiplied by ω_r also, for the case of the sphere. This supplies a useful check.

The numerical management of these formulæ for actual systems is dealt with later. I shall now consider their analytical and geometrical properties.

If

$$(b, \beta)\begin{Bmatrix}G+\delta G, & H+\delta H\\ K+\delta K, & L+\delta L\end{Bmatrix} = (b', \beta'),$$

and δG, ... are quadratic functions of b, c, β, γ, with the symmetries implied in the forms above, we may put

$$\left.\begin{aligned}\delta G &= \tfrac{1}{2}\{\delta_1 G (b^2+c^2) + 2\delta_2 G (b\beta + c\gamma) + \delta_3 G (\beta^2+\gamma^2)\},\\ \delta H &= \tfrac{1}{2}\{\delta_1 H (b^2+c^2) + 2\delta_2 H (b\beta + c\gamma) + \delta_3 H (\beta^2+\gamma^2)\},\\ \delta K &= \tfrac{1}{2}\{\delta_1 K (b^2+c^2) + 2\delta_2 K (b\beta + c\gamma) + \delta_3 K (\beta^2+\gamma^2)\},\\ \delta L &= \tfrac{1}{2}\{\delta_1 L (b^2+c^2) + 2\delta_2 L (b\beta + c\gamma) + \delta_3 L (\beta^2+\gamma^2)\}.\end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (15)$$

In these expressions the values of $\delta_1 G, \ldots$ are not unrestricted. Thus, for example, the rays which originate in the point (b, c) must upon emergence be normal to a surface. Consider the conditions that

$$y' = \beta' x' + b', \qquad z' = \gamma' x' + c',$$

where b', β', c', γ' are functions of two variables β, γ, as above, should be normal to a surface.

If (x', y', z') is a point upon the surface, then we have for all directions upon the surface

$$dx' + \beta' dy' + \gamma' dz' = 0,$$

or, since x', y', z' are functions of β, γ only,

$$\frac{\partial x'}{\partial \beta} + \beta' \frac{\partial y'}{\delta \beta} + \gamma' \frac{\partial z'}{\partial \beta} = 0, \qquad \frac{\partial x'}{\partial \gamma} + \beta' \frac{\partial y'}{\partial \gamma} + \gamma' \frac{\partial z'}{\partial \gamma} = 0.$$

Also

$$y' = \beta' x' + b', \qquad z' = \gamma' x' + c'.$$

Therefore

$$\frac{\partial x'}{\partial \beta}(1+\beta'^2+\gamma'^2) + \beta'\left(x' \frac{\partial \beta'}{\partial \beta} + \frac{\partial b'}{\partial \beta}\right) + \gamma'\left(x' \frac{\partial \gamma'}{\partial \beta} + \frac{\partial c'}{\partial \beta}\right) = 0,$$

$$\frac{\partial x'}{\partial \gamma}(1+\beta'^2+\gamma'^2) + \beta'\left(x' \frac{\partial \beta'}{\partial \gamma} + \frac{\partial b'}{\partial \gamma}\right) + \gamma'\left(x' \frac{\partial \gamma'}{\partial \gamma} + \frac{\partial c'}{\partial \gamma}\right) = 0\ ;$$

or, say,

$$\frac{\partial}{\partial \beta}\{x'(1+\beta'^2+\gamma'^2)^{1/2}\} + \left(\beta' \frac{\partial b'}{\partial \beta} + \gamma' \frac{\partial c'}{\partial \beta}\right)(1+\beta'^2+\gamma'^2)^{-1/2} = 0,$$

$$\frac{\partial}{\partial \gamma}\{x'(1+\beta'^2+\gamma'^2)^{1/2}\} + \left(\beta' \frac{\partial b'}{\partial \gamma} + \gamma' \frac{\partial c'}{\partial \gamma}\right)(1+\beta'^2+\gamma'^2)^{-1/2} = 0,$$

so that the necessary and sufficient condition is

$$\frac{\partial\,\{b', \beta'/(1+\beta'^2+\gamma'^2)^{1/2}\}}{\partial\,(\beta, \gamma)} + \frac{\partial\,\{c', \gamma'/(1+\beta'^2+\gamma'^2)^{1/2}\}}{\partial\,(\beta, \gamma)} = 0.$$

Retaining only the terms of lowest order we have

$$\frac{\partial b'}{\partial \beta} = \mathrm{H} \qquad \frac{\partial b'}{\partial \gamma} = \delta_2 \mathrm{G}\,.\,bc + \delta_3 \mathrm{G}\,.\,b\gamma + \delta_2 \mathrm{H}\,.\,c\beta + \delta_3 \mathrm{H}\,.\,\beta\gamma,$$

$$\frac{\partial \beta'}{\partial \beta} = \mathrm{L} \qquad \frac{\partial \beta'}{\partial \gamma} = \delta_2 \mathrm{K}\,.\,bc + \delta_3 \mathrm{K}\,.\,b\gamma + \delta_2 \mathrm{L}\,.\,c\beta + \delta_3 \mathrm{L}\,.\,\beta\gamma,$$

$$\frac{\partial c'}{\partial \beta} = \delta_2 \mathrm{G}\,.\,bc + \delta_3 \mathrm{G}\,.\,c\beta + \delta_2 \mathrm{H}\,.\,b\gamma + \delta_3 \mathrm{H}\,.\,\beta\gamma \qquad \frac{\partial c'}{\partial \gamma} = \mathrm{H},$$

$$\frac{\partial \gamma'}{\partial \beta} = \delta_2 \mathrm{K}\,.\,bc + \delta_3 \mathrm{K}\,.\,c\beta + \delta_2 \mathrm{L}\,.\,b\gamma + \delta_3 \mathrm{L}\,.\,\beta\gamma \qquad \frac{\partial \gamma'}{\partial \gamma} = \mathrm{L},$$

whence the condition

$$\begin{vmatrix} H & \delta_3 G - \delta_2 H \\ L & \delta_3 K - \delta_2 L \end{vmatrix} (b\gamma - c\beta) = 0,$$

$$\frac{\delta_3 G - \delta_2 H}{H} = \frac{\delta_3 K - \delta_2 L}{L} = \mathfrak{P}, \text{ say.} \quad \ldots \ldots \quad (16)$$

Also, this result must remain valid if we pass the emergent beam through any further optical system. This is a step that must frequently be taken, and it will be convenient to write down generally the formulæ to which it gives rise.

If we have

$$\begin{Bmatrix} g+\delta g, & h+\delta h \\ k+\delta k, & l+\delta l \end{Bmatrix} \begin{Bmatrix} g'+\delta g', & h'+\delta h' \\ k'+\delta k', & l'+\delta l' \end{Bmatrix} = \begin{Bmatrix} G+\delta G, & H+\delta H \\ K+\delta K, & L+\delta L \end{Bmatrix},$$

and if

$$\delta g = \tfrac{1}{2}\{\delta_1 g\,(b^2+c^2) + 2\delta_2 g\,(b\beta+c\gamma) + \delta_3 g\,(\beta^2+\gamma^2)\}, \ldots,$$

$$\delta g' = \tfrac{1}{2}\{\delta_1 g'\,(b'^2+c'^2) + \ldots \},$$

$$= \tfrac{1}{2}\{\delta_1 g\,[(gb+h\beta)^2+(gc+h\gamma)^2] + \ldots \}, \ldots,$$

and, further,

$$\delta G = \tfrac{1}{2}\{\delta_1 G\,(b^2+c^2) + 2\delta_2 G\,(b\beta+c\gamma) + \delta_3 G\,(\beta^2+\gamma^2)\}, \ldots,$$

then the following formulæ result:—

$$\begin{aligned}
\delta_1 G &= g'\delta_1 g + h'\delta_1 k + g\,\{g^2\delta_1 g' + 2gk\delta_2 g' + k^2\delta_3 g'\} + k\,\{g^2\delta_1 h' + 2gk\delta_2 h' + k^2\delta_3 h'\},\\
\delta_1 H &= g'\delta_1 h + h'\delta_1 l + h\,\{\quad ibid. \quad\} + l\,\{\quad ibid. \quad\},\\
\delta_1 K &= k'\delta_1 g + l'\,\delta_1 k + g\,\{g^2\delta_1 k' + 2gk\delta_2 k' + k^2\delta_3 k'\} + k\,\{g^2\delta_1 l' + 2gk\delta_2 l' + k^2\delta_3 l'\},\\
\delta_1 L &= k'\delta_1 h + l'\,\delta_1 l + h\,\{\quad ibid. \quad\} + l\,\{\quad ibid. \quad\},
\end{aligned}$$

$$\begin{aligned}
\delta_2 G &= g'\delta_2 g + h'\delta_2 k + g\,\{gh\delta_1 g' + (gl+hk)\,\delta_2 g' + kl\delta_3 g'\} + k\,\{gh\delta_1 h' + (gl+hk)\,\delta_2 h' + kl\delta_3 h'\},\\
\delta_2 H &= g'\delta_2 h + h'\delta_2 l + h\,\{\quad ibid. \quad\} + l\,\{\quad ibid. \quad\},\\
\delta_2 K &= k'\delta_2 g + l'\,\delta_2 k + g\,\{gh\delta_1 k' + (gl+hk)\,\delta_2 k' + kl\delta_3 k'\} + k\,\{gh\delta_1 l' + (gl+hk)\,\delta_2 l' + kl\delta_3 l'\},\\
\delta_2 L &= k'\delta_2 h + l'\,\delta_2 l + h\,\{\quad ibid. \quad\} + l\,\{\quad ibid. \quad\},
\end{aligned}$$

$$\begin{aligned}
\delta_3 G &= g'\delta_3 g + h'\delta_3 k + g\,\{h^2\delta_1 g' + 2hl\delta_2 g' + l^2\delta_3 g'\} + k\,\{h^2\delta_1 h' + 2hl\delta_2 h' + l^2\delta_3 h'\},\\
\delta_3 H &= g'\delta_3 h + h'\delta_3 l + h\,\{\quad ibid. \quad\} + l\,\{\quad ibid. \quad\},\\
\delta_3 K &= k'\delta_3 g + l'\,\delta_3 k + g\,\{h^2\delta_1 k' + 2hl\delta_2 k' + l^2\delta_3 k'\} + k\,\{h^2\delta_1 l' + 2hl\delta_2 l' + l^2\delta_3 l'\},\\
\delta_3 L &= k'\delta_3 h + l'\,\delta_3 l + h\,\{\quad ibid. \quad\} + l\,\{\quad ibid. \quad\}.
\end{aligned}$$

These formulæ with

$$G = gg' + kh', \qquad H = hg' + lh',$$

$$K = gk' + kl', \qquad L = hk' + ll',$$

are of fundamental importance and cover all cases; they will be quoted as

. . . . (17)

Apply them to the equation (16); we have

$$\delta_3 G - \delta_2 H = g'(\delta_3 g - \delta_2 h) + h'(\delta_3 k - \delta_2 l) + hn(\delta_2 g' - \delta_1 h') + ln(\delta_3 g' - \delta_2 h'),$$
$$\delta_3 K - \delta_2 L = k'(\delta_3 g - \delta_2 h) + l'(\delta_3 k - \delta_2 l) + hn(\delta_2 k' - \delta_1 l') + ln(\delta_3 k' - \delta_2 l'),$$

where

$$n = gl - hk.$$

If we write in this

$$\frac{\delta_3 g - \delta_2 h}{h} = \frac{\delta_3 k - \delta_2 l}{l} = \mathfrak{p}, \qquad \frac{\delta_3 g' - \delta_2 h'}{h'} = \frac{\delta_3 k' - \delta_2 l'}{l'} = \mathfrak{p}',$$

we have

$$\delta_3 G - \delta_2 H = \mathfrak{p}(hg' + lh') + n\{h(\delta_2 g' - \delta_1 h') + \mathfrak{p}' lh'\},$$
$$= (\mathfrak{p} + n\mathfrak{p}')H + nh\{(\delta_2 g' - \delta_1 h') - \mathfrak{p}' g'\},$$
$$\delta_3 K - \delta_2 L = (\mathfrak{p} + n\mathfrak{p}')L + nh\{(\delta_2 k' - \delta_1 h') - \mathfrak{p}' k'\}.$$

In the same manner we find

$$\delta_2 G - \delta_1 H = (\mathfrak{p} + n\mathfrak{p}')G + nk\{(\delta_3 g' - \delta_2 h') - \mathfrak{p}' h'\},$$
$$\delta_2 K - \delta_1 L = (\mathfrak{p} + n\mathfrak{p}')K + nk\{(\delta_3 k' - \delta_2 l') - \mathfrak{p}' l'\}.$$

Compare these with (16) and remember that the two systems $(gh\ldots)$, $(g'h'\ldots)$ are arbitrary and independent of one another. Then we see that if for these systems

$$\frac{\delta_2 g - \delta_1 h}{g} = \frac{\delta_3 g - \delta_2 h}{h} = \frac{\delta_2 k - \delta_1 l}{k} = \frac{\delta_3 k - \delta_2 l}{l} = \mathfrak{p},$$
$$\frac{\delta_2 g' - \delta_1 h'}{g'} = \ldots = \ldots = \ldots = \mathfrak{p}',$$

then

$$\frac{\delta_2 G - \delta_1 H}{G} = \frac{\delta_3 G - \delta_2 H}{H} = \frac{\delta_2 K - \delta_1 L}{K} = \frac{\delta_3 K - \delta_2 L}{L} = \mathfrak{P} \quad . \quad . \quad . \quad . \quad (18)$$

where

$$\mathfrak{P} = \mathfrak{p} + n\mathfrak{p}'.$$

Now if we examine the case of the single surface, for which

$$\delta_1 g = (1-n)B^2, \quad \delta_2 g = 0, \quad \delta_3 g = 0, \qquad \delta_1 h = (1-n)B, \quad \delta_2 h = 0, \quad \delta_3 h = 0,$$
$$\delta_1 k = (1-n)(-\epsilon + n - n^2)B^3, \quad \delta_2 k = -n^2(1-n)B^2, \quad \delta_3 k = -n(1-n^2)B,$$
$$\delta_1 l = (1-n)(-1+n-n^2)B^2, \quad \delta_2 l = -n^2(1-n)B, \quad \delta_3 l = -n(1-n^2),$$

and

$$g = 1, \quad h = 0, \quad k = -(1-n)B, \quad l = n,$$

the conditions are fulfilled and we have

$$\mathfrak{p} = -(1-n)B = \mu\left(\frac{1}{\mu'} - \frac{1}{\mu}\right)B;$$

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for two surfaces B_0, B_2 with refractive indices, μ_{-1}, μ_1, μ_3, as in SEIDEL'S convenient notation

$$\mathfrak{p}+n\mathfrak{p}' = \mu_{-1}\left[\left(\frac{1}{\mu_1}-\frac{1}{\mu_{-1}}\right)B_0+\left(\frac{1}{\mu_3}-\frac{1}{\mu_1}\right)B_2\right];$$

and for any sequence of surfaces whatever

$$\mathfrak{P} = \mu_{-1}\Sigma\left(\frac{1}{\mu_{2r+1}}-\frac{1}{\mu_{2r-1}}\right)B_{2r}. \quad \quad (19)$$

This will be recognized as the expression which figures in the well-known "Petzval condition for flatness of field." It was given by PETZVAL without proof in 1843, and it is a comment upon the difficulty which the geometrical method finds in removing a condition that may have been tacitly introduced that its proper position has so far remained obscure. Its general geometrical implications will be considered later.

Besides the condition that the rays of any thin bundle should always be normal to a surface there is another general property to which they are subject in all systems. For normal systems in which we have stigmatic correspondence this is usually called the Helmholtz magnification theorem connecting the linear and angular magnifications. For aberrational systems it would at first appear as if both linear and angular magnifications lost their meaning, but I have succeeded in generalizing the theorem in the paper already referred to.* In the first place focal lines in the original system are shown to correspond one to one and not pair to pair with focal lines in the emergent system; and rays which issue from any point in a focal line in a plane perpendicular to that line lie in a plane in the emergent system perpendicular to the conjugate focal line which they meet in a point. Such planes are called planes of correspondence. The behaviour of any ray may be traced through the behaviour of its projections upon the planes of correspondence. The separation of two parallel focal lines compared with the separation of their two conjugates preserves the idea of linear magnification and the angles in the planes of correspondence that of angular magnification. Then if a is the separation of two focal lines which lie parallel to one another in a plane perpendicular to an original ray at any point and a' that of their two conjugates, and if α is the angle between two rays issuing from one of these lines in a plane of correspondence perpendicular to both and α' the angle between the same rays on emergence, it is proved that

$$\mu a\alpha = \mu' a'\alpha'.$$

This is completely general. Now return to the case of surfaces centred upon an axis. It is clear that for any point off the axis, say the point $(0, b, 0)$, one of the planes of correspondence, is the meridianal plane passing through the axis and the point itself.

* 'Proc. L. M. S.,' vol. 29, p. 70.

Now consider the substitution

$$b' = (G+\delta G)\, b + (H+\delta H)\, \beta,$$
$$\beta' = (K+\delta K)\, b + (L+\delta L)\, \beta.$$

Then if we shift the origin in the emergent system to d' the first will read

$$b' = \{(G+\delta G) + d'(K+\delta K)\}\, b + \{(H+\delta H) + d'(L+\delta L)\}\, \beta.$$

Choose d' so as to make the coefficient of β zero; then

$$b' = \{(G+\delta G) - (K+\delta K)(H+\delta H)/(L+\delta L)\}\, b,$$

and the coefficient

$$G+\delta G - (K+\delta K)(H+\delta H)/(L+\delta L)$$

is the linear magnification for narrow pencils emerging in the general direction (β) from the point $(0, b, 0)$ in the meridianal plane. Again from

$$\beta'_1 - \beta'_2 = (L+\delta L)(\beta_1 - \beta_2),$$

the angular magnification for the same is $L+\delta L$.

Hence $(G+\delta G)(L+\delta L) - (H+\delta H)(K+\delta K)$ is equal to the ratio of the effective refractive indices. But we have seen on p. 157 that the change of ray effected by an aberrational system is equivalent to the use of refractive indices $\mu(1-\frac{1}{2}\beta^2)$, $\mu'(1-\frac{1}{2}\beta'^2)$, ... throughout. So that the expression above is equal to

$$\frac{\mu}{\mu'}(1-\tfrac{1}{2}\beta^2+\tfrac{1}{2}\beta'^2),$$

where β' may be taken as the final value of β after any number of transformations; or equal to

$$N\{1+\tfrac{1}{2}(Kb+L\beta)^2-\tfrac{1}{2}\beta^2\}.$$

Identifying term by term with the expression above we have the relations

$$\begin{aligned} \delta_1 N &= G\delta_1 L + L\delta_1 G - H\delta_1 K - K\delta_1 H = K^2 N, \\ \delta_2 N &= G\delta_2 L + L\delta_2 G - H\delta_2 K - K\delta_2 H = KLN, \qquad \qquad (20) \\ \delta_3 N &= G\delta_3 L + L\delta_3 G - H\delta_3 K - K\delta_3 L = (L^2-1)\, N. \end{aligned}$$

The relations (20) may also be proved from a sequence formula out of the equation (17); thus

$$\begin{aligned} \frac{\delta_1 N}{N} &= \frac{\delta_1 n}{n} + \left\{g^2 \frac{\delta_1 n'}{n'} + 2gk\frac{\delta_2 n'}{n'} + k^2\frac{\delta_3 n'}{n'}\right\}, \\ \frac{\delta_2 N}{N} &= \frac{\delta_2 n}{n} + \left\{gh \frac{\delta_1 n'}{n'} + (gl+hk)\frac{\delta_2 n'}{n'} + kl\frac{\delta_3 n'}{n'}\right\}, \qquad \qquad (21) \\ \frac{\delta_3 N}{N} &= \frac{\delta_3 n}{n} + \left\{h^2 \frac{\delta_1 n'}{n'} + 2hl\frac{\delta_2 n'}{n'} + l^2\frac{\delta_3 n'}{n'}\right\}; \end{aligned}$$

in fact, it was by such a method that I found them; but their real significance is contained in the proof given above.

We have thus found among the twelve aberrational coefficients six relations which may be expressed in terms only of the focal length and other cardinal elements of the normal system, or seven, if we include the Petzval expression as of that class.

Let us consider next what geometrical description can be given of the occurrence or absence of the twelve coefficients. It must be remembered that for different choice of origins the coefficients do not preserve an identity. Thus if we shift O, the original origin to the point $(-d, o, o)$, the new set—$\delta_1 G, \ldots$—is given in terms of the old set—$\delta_1 g', \ldots$—by writing in the equations of p. 160.

$$\delta_1 g = \delta_1 h = \ldots = 0$$

and

$$g = 1, \qquad h = d, \qquad k = 0, \qquad l = 1;$$

and if, on the other hand, we shift the emergent origin to d', we have $\delta_1 G, \ldots$ connected with $\delta_1 g, \ldots$, which now figures as the old set, as if in the same equations we wrote

$$\delta_1 g' = \delta_1 h' = \ldots = 0,$$

$$g' = 1, \qquad h' = d', \qquad k' = 0, \qquad l' = 1,$$

and in the event of both these changes being made a system $(g+\delta g, \ldots)$ is transformed into $(G+\delta G, \ldots)$, where

$$G = g + d'k, \qquad H = h + dg + d'l + dd'k,$$

$$K = k, \qquad L = l + dk;$$

$$\left.\begin{aligned}
\delta_1 G &= \delta_1 g + d'\delta_1 k, \\
\delta_1 H &= \delta_1 h + d\delta_1 g + d'(\delta_1 l + d\delta_1 k), \\
\delta_1 K &= \delta_1 k, \\
\delta_1 L &= \delta_1 l + d\delta_1 k, \\
\delta_2 G &= \delta_2 g + d\delta_1 g + d'\delta_2 K, \\
\delta_2 H &= \delta_2 h + d(\delta_1 h + \delta_2 g) + d^2\delta_1 g + d'\delta_2 L, \\
\delta_2 K &= \delta_2 k + d\delta_1 k, \\
\delta_2 L &= \delta_2 l + d(\delta_1 l + \delta_2 k) + d^2\delta_1 k, \\
\delta_3 G &= \delta_3 g + 2d\delta_2 g + d^2\delta_1 g + d'\delta_3 K, \\
\delta_3 H &= \delta_3 h + d(2\delta_2 h + \delta_3 g) + d^2(\delta_1 h + 2\delta_2 g) + d^3\delta_1 g + d'\delta_3 L, \\
\delta_3 K &= \delta_3 k + 2d\delta_2 k + d^2\delta_1 k, \\
\delta_3 L &= \delta_3 l + d(2\delta_2 l + \delta_3 k) + d^2(\delta_1 l + 2\delta_2 k) + d^3\delta_1 k.
\end{aligned}\right\} \quad \ldots \quad (22)$$

But let us defer discussion of these and examine two particular cases of special importance, namely, let us assign meanings to δ_1G, ... : (1) where the emergent origin is the principal focus, so that G = 0, and therefore $\delta_2\mathrm{G} = \delta_1\mathrm{H}$, and (2) where the original and emergent origins are conjugate, so that H = 0, and therefore $\delta_3\mathrm{G} = \delta_2\mathrm{H}$. In the former the original origin may be anywhere, but may conveniently be supposed to lie at the tangent plane to the first refracting surface. The original rays are in constant direction, so that we may take

$$\beta = const., \qquad \gamma = 0, \quad \text{and, say,} \quad b = d\cos\phi, \qquad c = d\sin\phi.$$

Then if we receive the emergent ray on the plane parallel to $\mathrm{O}'y'z'$ which passes through a point slightly removed from O′, say at

$$x' = \delta f',$$

and it cuts this plane at $y' = b' + \delta b'$, $z' = c' + \delta c'$, we have

$$b' + \delta b' = [* + \mathrm{K}\delta f']\,b + [\mathrm{H} + \mathrm{L}\delta f']\,\beta$$
$$+ \tfrac{1}{2}d\cos\phi\,[d^2\delta_1\mathrm{G} + 2d\beta\cos\phi\,\delta_2\mathrm{G} + \beta^2\delta_3\mathrm{G}] + \tfrac{1}{2}\beta\,[d^2\delta_1\mathrm{H} + 2d\beta\cos\phi\,\delta_2\mathrm{H} + \beta^2\delta_3\mathrm{H}],$$

$$c' + \delta c' = [* + \mathrm{K}\delta f']\,c$$
$$+ \tfrac{1}{2}d\sin\phi\,[d^2\delta_1\mathrm{G} + 2d\beta\cos\phi\,\delta_2\mathrm{G} + \beta^2\delta_3\mathrm{G}]. \qquad (23)$$

Let us take

$$b' = (\mathrm{H} + \mathrm{L}\delta f')\,\beta, \qquad c' = 0,$$

so that

$$\delta b' = \tfrac{1}{2}\beta\,[d^2(\delta_1\mathrm{H} + \delta_2\mathrm{G}) + \beta^2\delta_3\mathrm{H}]$$
$$+ \cos\phi d\,[\mathrm{K}\delta f' + \tfrac{1}{2}d^2\delta_1\mathrm{G} + \tfrac{1}{2}\beta^2(\delta_3\mathrm{G} + 2\delta_2\mathrm{H})]$$
$$+ \cos 2\phi d^2\,[\tfrac{1}{2}\beta\delta_2\mathrm{G}],$$

$$\delta c' = \sin\phi d\,[\mathrm{K}\delta f' + \tfrac{1}{2}d^2\delta_1\mathrm{G} + \tfrac{1}{2}\beta^2\delta_3\mathrm{G}]$$
$$+ \sin 2\phi d^2\,[\tfrac{1}{2}\beta\delta_2\mathrm{G}]. \qquad (24)$$

These express the amounts by which the aberrations disturb the ray from its normal focus. Consider the lines in turn and examine their significance when the original ray traces out a circle $d = const.$

The terms

$$\tfrac{1}{2}\beta\,[d^2(\delta_1\mathrm{H} + \delta_2\mathrm{G}) + \beta^2\delta_3\mathrm{H}] \quad \text{or} \quad \tfrac{1}{2}\beta\,[2d^2\delta_2\mathrm{G} + \beta^2\delta_3\mathrm{H}]$$

give a fixed point. It may be considered as adding to the focal length

$$\mathrm{H} + \mathrm{L}\delta f'$$

the terms

$$d^2\delta_2\mathrm{G} + \tfrac{1}{2}\beta^2\delta_3\mathrm{H},$$

of which the former may be called the *comatic increase of focal length* and the latter the *distortional increase of focal length*, since these are evidently their characters.

The two terms

$$\cos\phi d\,[K\delta f'+\tfrac{1}{2}d^2\delta_1 G+\tfrac{1}{2}\beta^2(\delta_3 G+2\delta_2 H)],\qquad \sin\phi d\,[K\delta f'+\tfrac{1}{2}d^2\delta_1 G+\tfrac{1}{2}\beta^2\delta_3 G]$$

represent an ellipse which may be varied by choosing $\delta f'$ at different values. If we take

$$K\delta f'+\tfrac{1}{2}d^2\delta_1 G+\tfrac{1}{2}\beta^2(\delta_3 G+2\delta_2 H) = 0$$

the ellipse becomes the primary focal line; if we take

$$K\delta f'+\tfrac{1}{2}d^2\delta_1 G+\tfrac{1}{2}\beta^2\delta_3 G = 0$$

it becomes the secondary focal line, in advance of the primary line by the amount $\beta^2\delta_2 H/K$. Generally I shall call it the *focal ellipse* and, as a rule, shall take

$$K\delta f'+\tfrac{1}{2}d^2\delta_1 G+\tfrac{1}{2}\beta^2(\delta_3 G+\delta_2 H) = 0,$$

which gives the *focal circle*

$$d\cos\phi\,[\tfrac{1}{2}\beta^2\delta_2 H],\qquad -d\sin\phi\,[\tfrac{1}{2}\beta^2\delta_2 H]$$

situated midway between the focal lines. This circle is described backwards as the original circle $d = const.$ is described forwards.

Finally the terms

$$d^2\cos 2\phi\,[\tfrac{1}{2}\beta\delta_2 G],\qquad d^2\sin 2\phi\,[\tfrac{1}{2}\beta\delta_2 G]$$

give another circle which I shall call the *comatic circle;* its radius $= \tfrac{1}{2}\beta \times$ *comatic increase of focal length*, so that they vanish together. As the original circle $d = const.$ is described once, forward, it is described twice, forward, each point upon it corresponding to two diametrically opposite points of the original circle.

Consider the focal circle and the comatic circle simultaneously; we may take

$$l\cos\phi+m\cos 2\phi,\qquad -l\sin\phi+m\sin 2\phi,$$

where

$$l = \tfrac{1}{2}d\beta^2\delta_2 H,\qquad m = \tfrac{1}{2}d^2\beta\delta_2 G;$$

this is a trochoidal curve, which becomes a three cusped hypocycloid for $l = 2m$ and goes through the types illustrated below for different values of β/d. For a given value of β all these types are present for different values of d, and are described about different centres owing to the comatic increase of focal length. These facts are well known in particular instances, and even experimentally, but as far as I can find they have not hitherto been expressly demonstrated generally.

The plane at which these phenomena are found is taken at $x' = \delta f'$, where

$$\delta f'/f' = -K\delta f' = \tfrac{1}{2}d^2\delta_1 G+\tfrac{1}{2}\beta^2(\delta_3 G+\delta_2 H).$$

The part depending upon d^2 represents the spherical aberration; the part depending upon β^2, if it were present alone, would indicate that the images, if we can so call them, were found upon a sphere of curvature $-K(\delta_3 G+\delta_2 H)$; the

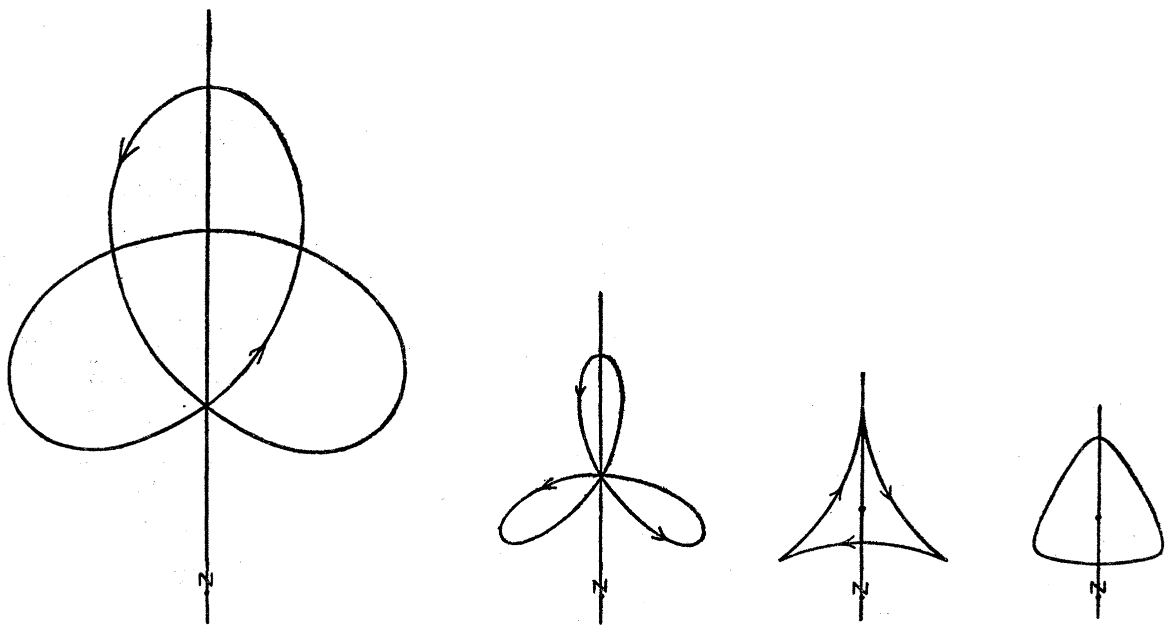

Fig. 5.

corresponding expressions for the primary focal line in place of the focal circle would be $-K(\delta_3 G+2\delta_2 H)$ and for the secondary focal line $-K\delta_3 G$. These expressions are positive when the sphere is convex to the incident rays.

The angular values of the radii of the focal circle and the comatic circle are respectively

$$\overset{(1)}{206265'' \times \tfrac{1}{2}\delta_2 H \,.\, d\beta^2/f'} \quad \text{and} \quad \overset{(2)}{206265'' \times \tfrac{1}{2}\delta_2 G \,.\, d^2\beta/f'}.$$

Thus with increasing aperture (d) the focal radius increases with the first power and the comatic radius with the square, while with increasing breadth of field (β) the focal radius increases with the square and the comatic radius with the first power. To fix ideas we may consider the case of the parabolic reflector; here, as shown on p. 156,

$$\delta_1 G = 0, \quad \delta_2 G = +1/2f', \quad \delta_3 G = 0, \quad \delta_1 H = +1/2f', \quad \delta_2 H = -1, \quad \delta_3 H = 0.$$

Hence spherical aberration and distortion are absent. $\delta_3 G = 0$ implies that the secondary focal line lies in the normal focal plane, while the focal circle lies upon a surface of curvature $1/2f'$; and for the effects of astigmatism and coma we have the following table for different apertures and fields* :—

* *Cf.* POOR, 'Astrophysical Journal,' VII. (1898), p. 121.

	$d = f/30$.				$d = f/20$.				$d = f/10$.			
β.	15′.	30′.	45′.	60′.	15′.	30′.	45′.	60′.	15′.	30′.	45′.	60′.
Focal radius .	0″·06	0″·26	0″·59	1″·05	0″·10	0″·40	0″·89	1″·57	0″·19	0″·79	1″·77	3″·14
Comatic radius	0″·25	0″·50	0″·75	1″·00	0″·56	1″·13	1″·69	2″·25	2″·25	4″·50	6″·75	9″·00
Comatic magnification .	1·0005	1·0005	1·0005	1·0005	1·0012	1·0012	1·0012	1·0012	1·0050	1·0050	1·0050	1·0050
Comatic displacement .	0″·50	1″·00	1″·50	2″·00	1″·13	2″·25	3″·38	4″·50	4″·50	9″·00	13″·50	18″·00

The last two lines measure the same thing, the fourth representing $\frac{1}{2}(\delta_1 H + \delta_2 G)\, d^2\beta$ in angle, and the third $1 + \frac{1}{2}(\delta_1 H + \delta_2 G)\, d^2\beta/\beta f'$, and owing to the relation $\delta_2 G = \delta_1 H$ the second line contains quantities one-half that of the fourth. Since these aberrations stand uncompensated, it is clear that the statement often made that the reflector has a very limited field is fully borne out, especially when, as is often the case, the ratio of semi-aperture to focal length is so great as 1/10. In this case, at only 30′ from the centre of the field, the light which comes from the outermost zone of the mirror would be spread around a little ring which was nearly a circle of 10″ diameter, having its centre 9″ from the correct normal position for the image.

Turn now to the other case which was proposed for discussion on p. 165, namely, where the original O, O′ are conjugate foci, so that $H = 0$ and $\delta_3 G = \delta_2 H$. We have to study the delineation of any point in Oyz upon the plane $O'y'z'$ or planes close to it. We may take the point $b = const.$, $c = 0$, and then make β, γ vary, so that, *e.g.*, for $\beta^2 + \gamma^2 = const.$ the ray through the point $(b, 0)$ describes a cone with axis parallel to Ox. Let us put

$$\beta = \theta \cos \psi, \qquad \gamma = \theta \sin \psi,$$

and we have, at the plane $x' = \delta f'$,

$$b' + \delta b' = [G + K\delta f']\, b + [\ * \ + L\delta f']\, \beta$$
$$+ \tfrac{1}{2} b [b^2 \delta_1 G + 2b\theta \cos\psi\, \delta_2 G + \theta^2 \delta_3 G] + \tfrac{1}{2}\theta \cos\psi\, [b^2 \delta_1 H + 2b\theta \cos\psi\, \delta_2 H + \theta^2 \delta_3 H],$$

$$c' + \delta c' = \qquad [\ * \ + L\delta f']\, \gamma$$
$$+ \qquad * \qquad + \tfrac{1}{2}\theta \sin\psi\, [b^2 \delta_1 H + 2b\theta \cos\psi\, \delta_2 H + \theta^2 \delta_3 H]. \quad (25)$$

Compare these with the expressions (23) of p. 165, and we see that they run upon exactly the same model, but with a change of *rôle* in which we replace

$$\delta_1 G, \quad \delta_2 G, \quad \delta_3 G, \quad \delta_1 H, \quad \delta_2 H, \quad \delta_3 H, \qquad d, \quad \phi, \quad \beta,$$

by

$$\delta_3 H, \quad \delta_2 H, \quad \delta_1 H, \quad \delta_3 G, \quad \delta_2 G, \quad \delta_1 G, \qquad \theta, \quad \psi, \quad b.$$

Hence it is unnecessary to work out the expressions for focal lines and the rest afresh since they can all be inferred without other change from what has already been given.

SEIDEL'S five conditions are usually taken as the standard form for the conditions of existence of a correct normal image. We may follow these and express them in terms of the aberrational coefficients, proceeding *pari passu* with the two cases :—

	O′ principal focus.		O, O′ conjugate foci.
(1) Absence of spherical aberration .	$\delta_1 G = 0$	or	$\delta_3 H = 0$
(2) Absence of coma	$\delta_2 G = \delta_1 H = 0$	,,	$\delta_2 H = \delta_3 G = 0$
(3) Absence of astigmatism	$\delta_2 H = 0$	,,	$\delta_2 G = 0$
(4) Absence of distortion	$\delta_3 H = 0$	,,	$\delta_1 G = 0$
(5) A flat field, when (2) and (3) are satisfied	$\delta_3 G = 0$	,,	$\delta_1 H = 0$. (26)

It is of interest to consider the position occupied by the well-known conditions usually quoted as "PETZVAL'S condition for flatness of field," and "ABBE'S sine-condition."

PETZVAL'S condition, or $\mathfrak{P} = 0$, we see from (18) to imply

$$\delta_2 G - \delta_1 H = 0, \quad \delta_3 G - \delta_2 H = 0, \quad \delta_2 K - \delta_1 L = 0, \quad \delta_3 K - \delta_2 L = 0,$$

or, what is the same thing, simply

$$\delta_2 G = \delta_1 H \qquad \text{and} \qquad \delta_3 G = \delta_2 H$$

at all distances along $O'x'$.

If we confine attention to the two cases above, we see that in the first, where the emergent origin is the principal focus, $G = 0$, and therefore $\delta_2 G = \delta_1 H$ without the intervention of $\mathfrak{P} = 0$, and similarly in the second, when the emergent and original origins are conjugate normal foci, $H = 0$, and therefore $\delta_3 G = \delta_2 H$; the interpretation of these is the same, namely, that the comatic displacement is twice the comatic radius—"comatic displacement" being used to denote the expression $\frac{1}{2}d^2\beta(\delta_1 H + \delta_2 G)$ as on p. 168—a well-known fact, usually put in the form that in the absence of astigmatism the successive comatic circles have two common tangents inclined to one another at 60 degrees. The other term remains as the true content of PETZVAL'S condition. Its interpretation may be put in different forms; as, apart from spherical aberration, at the normal focal plane of any image, the longitudinal axis of the focal ellipse is three times its transverse axis, which is an interpretation of the expressions of p. 166, for $\delta f' = 0$,

$$\delta b' = \ldots \cos\phi d\left[\tfrac{1}{2}d^2\delta_1 G + \tfrac{1}{2}\beta^2(\delta_3 G + 2\delta_2 H)\right], \qquad \delta c' = \ldots \sin\phi d\left[\tfrac{1}{2}d^2\delta_1 G + \tfrac{1}{2}\beta^2\delta_3 G\right],$$

of the first case, or the corresponding expression of the second case; or again, the distance of the focal circle beyond the normal focal plane is $2f'/d \times$ the radius of the

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focal circle. This shows the connection with flatness of field, but to restrict the reference to curvature of the field is a misconception of the significance of PETZVAL'S condition. If PETZVAL'S condition holds, the result stated above is true for all normal image planes.

ABBE'S sine-condition for absence of coma states that if the magnification produced by rays passing between two conjugate foci through all zones is the same, the relation must hold

$$\sin\theta'/\sin\theta = const.,$$

where θ, θ' are the original and emergent inclinations of the ray to the axis. In our notation this would run

$$(\beta'-\tfrac{1}{2}\beta'^3)/(\beta-\tfrac{1}{2}\beta^3) = const.;$$

or, if

$$\beta' = (L+\delta L)\,\beta$$

it gives

$$\delta L/L = \tfrac{1}{2}(L^2-1)\,\beta^2;$$

now we have seen on p. 163 that the linear magnification is

$$(N+\delta N)/(L+\delta L)$$

and the condition this should be constant is

$$\delta L/L = \delta N/N,$$

and this, in accordance with p. 163, gives the conditions

$$\delta_1 L/L = K^2, \quad \delta_2 L/L = KL, \quad \delta_3 L/L = L^2-1.$$

To make these agree with the sine-condition we must take $b = 0$, so as to remove $\delta_1 L$, $\delta_2 L$ from the reckoning. We see then that such an assumption underlies the application of the sine-condition.

I shall next show how these formulæ may be applied to the numerical calculation of lenses. For this they are particularly appropriate if the calculations are made with any ordinary type of multiplying machine and not with logarithms. To the best of my judgment they appear to require a fraction only of the work involved in the equivalent complete trigonometrical calculation and, as will be shown, they are certainly not less accurate for telescopic object glasses. They show with remarkable clearness the contribution of each surface to each fault of the image. They also supply throughout their course a number of natural checks upon the computation which are searching and usually complete.

I shall take as my example the celebrated object glass of the Fraunhofer heliometer at Königsberg. This is a small lens of aperture 6·2 inches and focal length 101 inches which was constructed by FRAUNHOFER. BESSEL, in describing the heliometer and its corrections, with his customary masterly thoroughness, calculated this lens trigonometrically.* Later it was used as an illustration by SEIDEL,† who named his second condition the Fraunhofer condition, under the misapprehension that coma was effectively corrected in it.

In 1889, Dr. A. STEINHEIL gave a particularly thorough and instructive calculation of its field, and assigned first, the modifications of its curves necessary to remove a remaining trace of spherical aberration, and next, to correct the coma.‡ Finally, FINSTERWALDER recalculated SEIDEL'S sums for it, using the first corrected curves of STEINHEIL.§ In the following pages I shall first of all show how the calculations will run with my formulæ, and shall return to compare them with STEINHEIL'S results.

The data given by BESSEL are in "lines," of which 144 = 1 Bavarian foot. To render the arithmetical work more compact I have increased the unit to 1000 lines, which brings the measures of the radii of the surfaces and the focal length of the whole to the neighbourhood of a unit.

The radii and curvatures of the surfaces, with the spaces between them, are the following:—

$$\rho_0 = +\ {\cdot}838164,\qquad B_0 = +1{\cdot}193084,\qquad d_1 = {\cdot}006,$$
$$\rho_2 = -\ {\cdot}333768,\qquad B_2 = -2{\cdot}996093,\qquad d_3 = {\cdot}000,$$
$$\rho_4 = -\ {\cdot}340536,\qquad B_4 = -2{\cdot}936547,\qquad d_5 = {\cdot}004.$$
$$\rho_6 = -1{\cdot}172508,\qquad B_6 = -\ {\cdot}852873,$$

The semi-aperture is 0·0351.

The refractive indices he takes as

$$n = \mu_{-1}/\mu_1 = {\cdot}653966,\qquad 1/n = 1{\cdot}529130,$$
$$m = \mu_3/\mu_5 = {\cdot}610083,\qquad 1/m = 1{\cdot}639121.$$

We first form the normal scheme for the whole combination by writing down and combining the schemes that represent each surface and each space between two surfaces. To perform the step of combination

$$\begin{Bmatrix} g, & h \\ k, & l \end{Bmatrix} \begin{Bmatrix} \gamma, & \eta \\ \kappa, & \lambda \end{Bmatrix} = \begin{Bmatrix} \gamma g + \eta k, & \gamma h + \eta l \\ \kappa g + \lambda k, & \kappa h + \lambda l \end{Bmatrix}$$

—as to which it must be remembered that light passes through $\{g \ldots\}$ to reach $\{\gamma \ldots\}$—we set up the number γ as multiplier upon the multiplying machine, multiply it into g and h, and place the products as above, then set up η and multiply into k and l, placing the products as shown; this gives the top line of the combination completely; then set up κ, multiply into g and h, set up λ and multiply it into k and l.

* 'Untersuchungen,' Bd. I., p. 101.

† 'Astronomische Nachrichten,' No. 1029, p. 325.

‡ K. Bayer. Akad. d. Wiss., 'Sitzungberichte d. math.-phys. Classe,' Bd. XIX., Heft III., 1889.

§ K. Bayer. Akad. d. Wiss., 'Abhandlungen,' Bd. XVII., Abth. III.

This completes the step. The proper way to check a sequence of such combinations is to proceed first from left to right and then from right to left; the final results will confirm one another and check the whole calculation, and the individual schemes arrived at will give in succession the steps across the surface (0), across the surface (0) and the space (1), across 0, 1 and the surface 2, and so on; and in the reverse order across the surface 6, across 5 and 6, across 4, 5, and 6, and so on. All these will be required after. I shall indicate them with the signs $\vec{0}$, $\overrightarrow{01}$, ..., $\vec{6}$, $\overrightarrow{56}$, ...

For the individual surfaces the values of $k = -(1-n)\,B$ are

$$k_0 = -(1-n)\,B_0 = -\ \cdot 412848, \qquad k_2 = -(1-1/n)\,B_2 = -1\cdot 585323,$$
$$k_4 = -(1-m)\,B_4 = +1\cdot 145010, \qquad k_6 = -(1-1/m)\,B_6 = -\ \cdot 545089.$$

In the following arrangement the separate schemes are written in the middle column, that corresponding to $\vec{3}$ being omitted as nugatory; the calculations forwards are written on the left and those backwards on the right. The latter begin at the bottom and proceed upwards. Every figure used is recorded.

$\vec{0}$

			Scheme $\vec{0}$			$\overrightarrow{06}$	
					1·000712		
					− *4019*		
		1·000000	*		·996693	+ ·006366	
		− ·412848	·653966		− *·253990*		$\overrightarrow{06}$
					− *·629829*		
					− ·883819	+ ·997672	

$\vec{1}$

$\overrightarrow{01}$			Scheme $\vec{1}$			$\overrightarrow{16}$	
	1·000000					+ *·006004*	
	− *2477*					+ *3731*	
$\overrightarrow{01}$	·997523	+ ·003924	1·000000	·006000	+1·000712	+ ·009735	$\overrightarrow{16}$
	·412848	+ ·653966	*	1·000000		− *·001524*	
						+*1·527096*	
					− ·253990	+1·525572	

$\vec{2}$

$\overrightarrow{02}$			Scheme $\vec{2}$			$\overrightarrow{26}$	
					1·004580		
	·997523	+ ·003924			− *3868*		
	−*1·581396*	− *·006221*	1·000000	*	1·000712	+ ·003731	
$\overrightarrow{02}$	− *·631298*	+ *·999999*	−1·585323	+1·529130	+*1·329225*		$\overrightarrow{26}$
	−2·212694	+ ·993778			−*1·583215*		
					− ·253990	+1·527096	

$\vec{4}$

$\overrightarrow{04}$			Scheme $\vec{4}$			$\overrightarrow{46}$	
					1·000000		
	·997523	+ ·003924			+ *4580*		
	+*1·142174*	+ *·004493*	1·000000	*	1·004580	+ ·002440	
$\overrightarrow{04}$	−*1·349927*	+ *·606287*	+1·145010	+ ·610083	− *·545089*		$\overrightarrow{46}$
	− ·207753	+ ·610780			+*1·874314*		
					+1·329225	+ ·998670	

$$\overrightarrow{05}\left\{\begin{array}{rr} \cdot 997523 & +\ \cdot 003924 \\ -\quad 831 & +\quad 2443 \\ \hline \cdot 996692 & +\ \cdot 006367 \\ \\ -\ \cdot 207753 & +\ \cdot 610780 \end{array}\right\} \qquad \overrightarrow{5}\left\{\begin{array}{rr} 1\cdot 000000 & \cdot 004000 \\ * & 1\cdot 000000 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 000000 & +\ \cdot 004000 \\ & +1\cdot 639121 \\ & -\quad 2180 \\ \hline -\ \cdot 545089 & +1\cdot 636941 \end{array}\right\}\overrightarrow{56}$$

$$\overrightarrow{06}\left\{\begin{array}{rr} \cdot 996692 & +\ \cdot 006367 \\ \\ -\ \cdot 543286 & -\ \cdot 003471 \\ -\ \cdot 340532 & +1\cdot 001142 \\ \hline -\ \cdot 883818 & +\ \cdot 997671 \end{array}\right\} \qquad \overrightarrow{6}\left\{\begin{array}{rr} 1\cdot 000000 & * \\ -\ \cdot 545089 & 1\cdot 639121 \end{array}\right\}$$

It may be well to repeat what these schemes imply. Take the scheme $\overrightarrow{16}$. If b, β refer to any ray where it meets the tangent plane to the surface (0) after crossing that surface but before crossing the space 1, and b', β' refer to the same ray where it meets the tangent plane to the surface (6) after crossing that surface, then the scheme $\overrightarrow{16}$ states that

$$b' = +1\cdot 000712b + \cdot 009735\beta,$$
$$\beta' = -\ \cdot 253990b + 1\cdot 525572\beta,$$

and *mutatis mutandis* the same holds for c, γ, c', γ'.

We conclude, from the expressions on p. 151, that for the whole combination the cardinal features of the normal combination are given by

$$\mathrm{HF} = -1\cdot 131455 = -\mathrm{H'F'}, \quad 0\mathrm{F} = -1\cdot 128820, \quad 6\mathrm{F'} = +1\cdot 127712.$$

We next work out the schemes multiplied into each of the aberrational functions ω, as given in (11), p. 157. The schemes $\{g, h, \ldots\}$, $\{g', h', \ldots\}$ which respectively precede and follow the surface to which ω refers are read at once from the computation of the normal system just completed. As the surfaces are supposed to be spherical, we have $\epsilon = 1$. The general arrangement is as above, and the check consists in forming the combination first forwards and then backwards. Again every figure is shown, but now the decimal places may be reduced to five.

ω_0 :—

$$\left\{\begin{array}{rr} 1\cdot 00000 & +\ \cdot 83816 \\ -1\cdot 19308 & -1\cdot 00000 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00071 & +\ \cdot 83876 \\ -\ \cdot 01161 & -\ \cdot 00973 \\ \hline +\ \cdot 98910 & +\ \cdot 82903 \\ \\ -\ \cdot 28398 & -\quad 21288 \\ -1\cdot 82012 & -1\cdot 52556 \\ \hline -2\cdot 07410 & -1\cdot 73844 \end{array}\right\}$$

$$\left\{\begin{array}{rr} 1\cdot 00071 & +\ \cdot 83876 \\ -\ \cdot 01161 & -\ \cdot 00973 \\ \hline \cdot 98910 & +\ \cdot 82903 \\ \\ -\ \cdot 25398 & -\ \cdot 21288 \\ -1\cdot 82012 & -1\cdot 52556 \\ \hline -2\cdot 07410 & -1\cdot 73844 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00071 & +\ \cdot 00974 \\ -\ \cdot 25398 & +1\cdot 52556 \end{array}\right\}$$

ω_2 :—

$$\left\{\begin{array}{rr} \cdot 99752 & +\cdot 00392 \\ -\cdot 41285 & +\cdot 65397 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00938 & +\cdot 00397 \\ +\cdot 13944 & -\cdot 22087 \\ \hline 1\cdot 14882 & -\cdot 21690 \\ +4\cdot 31058 & +\cdot 01694 \\ +\cdot 59546 & -\cdot 94323 \\ \hline +4\cdot 90604 & -\cdot 92629 \end{array}\right\}$$

$$\left\{\begin{array}{rr} \cdot 99752 & +\cdot 00392 \\ +\cdot 13780 & -\cdot 21828 \\ \hline 1\cdot 13532 & -\cdot 21436 \\ +2\cdot 98866 & +\cdot 01174 \\ +\cdot 41285 & -\cdot 65397 \\ \hline +3\cdot 40151 & -\cdot 64223 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00000 & -\cdot 33377 \\ +2\cdot 99609 & -1\cdot 00000 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00458 & -\cdot 33530 \\ +\quad 731 & -\quad 244 \\ \hline 1\cdot 01189 & -\cdot 33774 \\ +1\cdot 32923 & -\cdot 44365 \\ +2\cdot 99208 & -\cdot 99866 \\ \hline +4\cdot 32131 & -1\cdot 44231 \end{array}\right\}$$

$$\left\{\begin{array}{rr} +1\cdot 14052 & -\cdot 21534 \\ +\cdot 00830 & -\cdot 00157 \\ \hline 1\cdot 14882 & -\cdot 21691 \\ +1\cdot 50909 & -\cdot 28493 \\ +3\cdot 39694 & -\cdot 64137 \\ \hline +4\cdot 90603 & -\cdot 92630 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00458 & +\cdot 00244 \\ +1\cdot 32923 & +\cdot 99867 \end{array}\right\}$$

ω_4 :—

$$\left\{\begin{array}{rr} \cdot 99752 & +\cdot 00392 \\ -2\cdot 21269 & +\cdot 99379 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00924 & +\cdot 00397 \\ +\cdot 76236 & -\cdot 34240 \\ \hline 1\cdot 77160 & -\cdot 33843 \\ +4\cdot 25130 & +\cdot 01671 \\ +3\cdot 21132 & -1\cdot 44231 \\ \hline +7\cdot 46262 & -1\cdot 42560 \end{array}\right\}$$

$$\left\{\begin{array}{rr} +\cdot 99752 & +\cdot 00392 \\ +\cdot 75351 & -\cdot 33843 \\ \hline 1\cdot 75103 & -\cdot 33451 \\ +2\cdot 92927 & +\cdot 01151 \\ +2\cdot 21269 & -\cdot 99379 \\ \hline +5\cdot 14196 & -\cdot 98228 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00000 & -\cdot 34054 \\ +2\cdot 93655 & -1\cdot 00000 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00000 & -\cdot 34054 \\ +\cdot 01175 & -\cdot 00400 \\ \hline 1\cdot 01175 & -\cdot 34454 \\ -\cdot 54509 & +\cdot 18562 \\ +4\cdot 80696 & -1\cdot 63694 \\ \hline +4\cdot 26187 & -1\cdot 45132 \end{array}\right\}$$

$$\left\{\begin{array}{rr} +1\cdot 75103 & -\cdot 33451 \\ \cdot 02057 & -\cdot 00393 \\ \hline +1\cdot 77160 & -\cdot 33844 \\ -\cdot 95447 & +\cdot 18234 \\ +8\cdot 41708 & -1\cdot 60793 \\ \hline +7\cdot 46261 & -1\cdot 42559 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00000 & +\cdot 00400 \\ -\cdot 54509 & +1\cdot 63694 \end{array}\right\}$$

ω_6 :—

$$\left\{\begin{array}{rr} \cdot 99669 & +\ \cdot 00636 \\ -\ \cdot 20775 & +\ \cdot 61078 \end{array}\right\} \qquad \left\{\begin{array}{rr} +\ \mathit{\cdot 99669} & +\ \mathit{\cdot 00636} \\ +\ \mathit{\cdot 24359} & -\ \mathit{\cdot 71615} \\ \hline +1\cdot 24028 & -\ \cdot 70979 \\ \\ +\ \mathit{\cdot 85005} & +\ \mathit{\cdot 00542} \\ +\ \mathit{\cdot 20775} & -\ \mathit{\cdot 61078} \\ \hline +1\cdot 05780 & -\ \cdot 60536 \end{array}\right\}$$

$$\left\{\begin{array}{rr} +\ \mathit{\cdot 99669} & +\ \mathit{\cdot 00636} \\ +\ \mathit{\cdot 24359} & -\ \mathit{\cdot 71615} \\ \hline 1\cdot 24028 & -\ \cdot 70979 \\ \\ +\ \mathit{\cdot 85005} & +\ \mathit{\cdot 00542} \\ +\ \mathit{\cdot 20775} & -\ \mathit{\cdot 61078} \\ \hline +1\cdot 05780 & -\ \cdot 60536 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00000 & -1\cdot 17251 \\ +\ \cdot 85287 & -1\cdot 00000 \end{array}\right\}$$

In the case of the first and last, the combination consisting of only two terms, the check calculation is a mere duplicate, and is, therefore, less searching than the others. The signs, in particular, should be examined to guard against a double error.

We next form the corresponding schemes for the function ψ, again in accordance with the formulæ (11). Owing to the occurrence of two zeroes in the scheme at the surface the calculation is somewhat simpler.

ψ_0 :—

$$\left\{\begin{array}{rr} * & * \\ +1\cdot 19308 & +1\cdot 00000 \end{array}\right\} \qquad \left\{\begin{array}{rr} +\ \cdot 01161 & +\ \cdot 00973 \\ +1\cdot 82012 & +1\cdot 52556 \end{array}\right\}$$

$$\left\{\begin{array}{rr} +\ \cdot 01161 & +\ \cdot 00973 \\ +1\cdot 82012 & +1\cdot 52556 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00071 & +\ \cdot 00973 \\ -\ \cdot 25398 & +1\cdot 52556 \end{array}\right\}$$

ψ_2 :—

$$\left\{\begin{array}{rr} \cdot 99752 & +\ \cdot 00392 \\ -\ \cdot 41285 & +\ \cdot 65397 \end{array}\right\} \qquad \left\{\begin{array}{rr} -\ \mathit{\cdot 00729} & -\ \mathit{\cdot 00003} \\ -\ \mathit{\cdot 00101} & +\ \mathit{\cdot 00160} \\ \hline -\ \cdot 00830 & +\ \cdot 00157 \\ \\ -\mathit{2\cdot 98466} & -\ \mathit{\cdot 01173} \\ -\ \mathit{\cdot 41230} & +\ \mathit{\cdot 65309} \\ \hline -3\cdot 39696 & +\ \cdot 64136 \end{array}\right\}$$

$$\left\{\begin{array}{rr} * & * \\ -\mathit{2\cdot 98866} & -\ \mathit{\cdot 01174} \\ -\ \mathit{\cdot 41285} & +\ \mathit{\cdot 65397} \\ \hline -3\cdot 40151 & +\ \cdot 64223 \end{array}\right\} \qquad \left\{\begin{array}{rr} * & * \\ -2\cdot 99609 & +1\cdot 00000 \end{array}\right\} \qquad \left\{\begin{array}{rr} -\ \cdot 00731 & +\ \cdot 00244 \\ -2\cdot 99208 & +\ \cdot 99866 \end{array}\right\}$$

$$\left\{\begin{array}{rr} -\ \cdot 00830 & +\ \cdot 00157 \\ -3\cdot 39694 & +\ \cdot 64137 \end{array}\right\} \qquad \left\{\begin{array}{rr} +1\cdot 00458 & +\ \cdot 00244 \\ +1\cdot 32922 & +\ \cdot 99866 \end{array}\right\}$$

ψ_4 :—

$$\left\{\begin{array}{rr} \cdot 99752 & +\ \cdot 00392 \\ -2\cdot 21269 & +\ \cdot 99379 \end{array}\right\} \qquad \left\{\begin{array}{rr} -\ \mathit{\cdot 01172} & -\ \mathit{\cdot 00005} \\ -\ \mathit{\cdot 00885} & +\ \mathit{\cdot 00398} \\ \hline -\ \cdot 02057 & +\ \cdot 00393 \\ -\mathit{4\cdot 79504} & -\ \mathit{\cdot 01884} \\ -\mathit{3\cdot 62204} & +\mathit{1\cdot 62677} \\ \hline -8\cdot 41708 & +1\cdot 60793 \end{array}\right\}$$

$$\left\{\begin{array}{rr} * & * \\ +5\cdot 14196 & -\ \cdot 98228 \end{array}\right\} \qquad \left\{\begin{array}{rr} * & * \\ -2\cdot 93655 & +1\cdot 00000 \end{array}\right\} \qquad \left\{\begin{array}{rr} -\ \cdot 01175 & +\ \cdot 00400 \\ -4\cdot 80696 & +1\cdot 63694 \end{array}\right\}$$

$$\left\{\begin{array}{rr} -\ \cdot 02057 & +\ \cdot 00393 \\ -8\cdot 41708 & +1\cdot 60793 \end{array}\right\} \qquad \left\{\begin{array}{rr} 1\cdot 00000 & +\ \cdot 00400 \\ -\ \cdot 54509 & +1\cdot 63694 \end{array}\right\}$$

ψ_6 :—

$$\left\{\begin{array}{rr} \cdot 99669 & +\ \cdot 00636 \\ -\ \cdot 20775 & +\ \cdot 61078 \end{array}\right\} \qquad \left\{\begin{array}{rr} * & * \\ -\ \mathit{\cdot 85005} & -\ \mathit{\cdot 00542} \\ -\ \mathit{\cdot 20775} & +\ \mathit{\cdot 61078} \\ \hline -1\cdot 05780 & +\ \cdot 60536 \end{array}\right\}$$

$$\left\{\begin{array}{rr} * & * \\ -\ \mathit{\cdot 85005} & -\ \mathit{\cdot 00542} \\ -\ \mathit{\cdot 20775} & +\ \mathit{\cdot 61078} \\ \hline -1\cdot 05780 & +\ \cdot 60536 \end{array}\right\} \qquad \left\{\begin{array}{rr} * & * \\ -\ \cdot 85287 & +1\cdot 00000 \end{array}\right\}$$

The schemes in the middle columns which precede and follow those belonging to the surface are the same for ω and ψ. They should be written down independently and read against one another to guard against errors of transcription.

Now, for any surface (r), in accordance with (14),

$$\omega_r = \tfrac{1}{2}(1-n_r)\,\mathrm{B}_r^2\,(b_r^2+c_r^2), \qquad \psi_r = \tfrac{1}{2}n_r\,(\beta'^2_r+\gamma'^2_r-\beta_r^2-\gamma_r^2)$$

where b_r, c_r, β_r, γ_r, β'_r, γ'_r are read from the normal schemes on pp. 172, 173 in terms of b_0, c_0, β_0, γ_0, the specification of the original ray. We now calculate these, noting that since b_r, β_r, β'_r run exclusively together, as do also c_r, γ_r, γ'_r, we need speak explicitly of the former only.

We require to form from the original data $\frac{1}{2}(1-n_r)\,\mathrm{B}_r^2$, that is $-k_r\times\frac{1}{2}\mathrm{B}_r$. There is no check upon these values and they should be examined with care like other fundamental numbers. Their values are shown in the table below.

We have then as regards ω :—

r.	b_r.		b_r^2.			$\frac{1}{2}(1-n_r)\,\mathrm{B}_r^2$.	ω_r.			
	Co-efficient b_0.	Co-efficient β_0.	Co-efficient b_0^2.	Co-efficient $b_0\beta_0$.	Co-efficient β_0^2.		Co-efficient b_0^2.	Co-efficient $b_0\beta_0$.	Co-efficient β_0^2.	Sum of co-efficients.
0	$+1\cdot 00000$	*	$1\cdot 00000$	*	*	$+\ \cdot 24630$	$+\ \cdot 24630$	*	*	$+\ \cdot 24630$
2	$+\ \cdot 99752$	$+\cdot 00392$	$+\ \cdot 99505$	$+\cdot 00782$	$+\cdot 00002$	$-2\cdot 37488$	$-2\cdot 36312$	$-\cdot 01857$	$-\cdot 00005$	$-2\cdot 38174$
4	*ibid.*	*ibid.*	*ibid.*	*ibid.*	*ibid.*	$+1\cdot 68120$	$+1\cdot 67288$	$+\cdot 01315$	$+\cdot 00003$	$+1\cdot 68606$
6	$+\ \cdot 99669$	$+\cdot 00636$	$+\ \cdot 99339$	$+\cdot 01268$	$+\cdot 00004$	$-\ \cdot 23245$	$-\ \cdot 23091$	$-\cdot 00295$	$-\cdot 00001$	$-\ \cdot 23387$

The best way of forming the square, *e.g.*, of b_2, is to set up ·99752 on the machine, multiply it into itself and into twice 392, then set up 392 and multiply it into itself and twice ·99752, when the agreement of the middle terms is a check upon the operations. The last column, "sum of coefficients in ω_r," will be used below as a check for future work. If necessary, it may be checked by the equation, *e.g.*, for ω_2,

$$-2{\cdot}38174 = -2{\cdot}37488 \times ({\cdot}99752 + {\cdot}00392)^2.$$

Next, for formation of ψ, arrange as below :—

r.	$\beta_r = \beta'_{r-2}$.		$\beta_r^2 = \beta'^2_{r-2}$.			$\beta'^2_r - \beta_r^2$.				$\frac{1}{2}n_r$.
	Coefficient b_0.	Coefficient β_0.	Coefficient b_0^2.	Coefficient $b_0\beta_0$.	Coefficient β_0^2.	Coefficient b_0^2.	Coefficient $b_0\beta_0$.	Coefficient β_0^2.	Sum of coefficients.	
0	*	+1·00000	*	*	1·00000	·17045	− ·53998	− ·57232	− ·94185	·32698
2	− ·41285	+ ·65397	+ ·17045	− ·53998	+ ·42768	4·72555	−3·85792	+ ·55994	+1·42757	·76457
4	−2·21269	+ ·99379	+4·89600	−4·39790	+ ·98762	−4·85284	+4·14412	− ·61457	−1·32329	·30504
6	− ·20775	+ ·61078	+ ·04316	− ·25378	+ ·37305	+ ·73798	−1·50974	+ ·62230	− ·14946	·81956
8	− ·88382	+ ·99767	+ ·78114	−1·76352	+ ·99535	+ ·78114	−1·76352	− ·00465	− ·98703	—

The last row under $\beta'^2_r - \beta_r^2$ is the sum of the numbers above it and is introduced as a check upon the subtractions; it is equal to $\beta'^2_6 - \beta_0^2$. The addition of mixed positive and negative numbers is best done with a machine. We now have

r.	ψ_r.			
	Coefficient b_0^2.	Coefficient $b_0\beta_0$.	Coefficient β_0^2.	Sum of coefficients.
0	+ ·05573	− ·17656	− ·18714	− ·30797
2	+3·61299	−2·94963	+ ·42811	+1·09147
4	−1·48032	+1·26413	− ·18747	− ·40366
6	+ ·60482	−1·23732	+ ·51001	− ·12249

The multiplication by $\frac{1}{2}n_r$ should be checked by the help of the column "sum of coefficients." It will be remembered that there is no check against setting up an erroneous multiplier for $\frac{1}{2}n_r$.

We are now ready to form δG, &c. Referring to (12) on p. 157 and the calculations above we have, for example,

$$\delta G = +{\cdot}98910\omega_0 + 1{\cdot}14882\omega_2 + 1{\cdot}77160\omega_4 + 1{\cdot}24028\omega_6 + {\cdot}01161\psi_0 - {\cdot}00830\psi_2 - 02057\psi_4 + {}^*,$$

the coefficient of ψ_2, for example, being the figure that stands in the place of G in the scheme formed on p. 175 for ψ_2; and similarly for δH, δK, δL. It is unnecessary to write them at length because they are shown in a more convenient place in the following table :—

	Co-efficient δG.	δG. Co-efficient b_0^2.	δG. Co-efficient $b_0\beta_0$.	δG. Co-efficient β_0^2.	Co-efficient δH.	δH. Co-efficient b_0^2.	δH. Co-efficient $b_0\beta_0$.	δH. Co-efficient β_0^2.	Co-efficient δK.	δK. Co-efficient b_0^2.	δK. Co-efficient $b_0\beta_0$.	δK. Co-efficient β_0^2.	Co-efficient δL.	δL. Co-efficient b_0^2.	δL. Co-efficient $b_0\beta_0$.	δL. Co-efficient β_0^2.	Sum of Co-efficient.
ω_0 . . .	+ ·98910	+ ·24362	*	*	+ ·82903	+ ·20419	*	*	−2·07410	− ·51085	*	*	−1·73844	− ·42818	*	*	− ·49122
ω_2 . . .	+1·14882	−2·71480	− ·02133	− ·00006	− ·21691	+ ·51258	+ ·00403	+ ·00001	+4·90603	−11·59354	− ·09110	− ·00025	− ·92630	+2·18896	+ ·01720	+ ·00005	−11·69825
ω_4 . . .	+1·77160	+2·96367	+ ·02330	+ ·00005	− ·33844	− ·56615	− ·00445	− ·00001	+7·46261	+12·48403	+ ·09813	+ ·00022	−1·42559	−2·38484	− ·01875	− ·00004	+12·59517
ω_6 . . .	+1·24028	− ·28639	− ·00366	− ·00001	− ·70979	+ ·16390	+ ·00209	+ ·00001	+1·05780	− ·24426	− ·00312	− ·00001	− ·60536	+ ·13978	+ ·00179	− ·00001	− ·22958
		+ ·20610	− ·00169	− ·00002		+ ·31452	+ ·00167	+ ·00001		+ ·13538	+ ·00391	− ·00004		− 48428	+ ·00024	·00000	+ ·17582 [+ ·17580]
ψ_0 . . .	+ ·01161	+ ·00065	− ·00205	− ·00217	+ ·00973	+ ·00054	− ·00172	− ·00182	+1·82012	+ ·10144	− ·32136	− ·34062	+1·52556	+ ·08502	− ·26935	− ·28549	− 1·03694
ψ_2 . . .	− ·00830	− ·02999	+ ·02448	− ·00355	+ ·00157	+ ·00567	− ·00463	+ ·00067	−3·39694	−12·27314	+10·01975	−1·45427	+ ·64137	+2·31726	−1·89180	+ ·27458	− 3·01498
ψ_4 . . .	− ·02057	+ ·03045	− ·02600	+ ·00386	+ ·00393	− ·00582	+ ·00497	− ·00074	−8·41708	+12·45997	−10·64028	+1·57795	+1·60793	−2·38025	+2·03263	− ·30144	+ 2·75530
ψ_6 . . .	*	*	*	*	*	*	*	*	−1·05780	− ·63978	+ 1·30884	− ·53949	+ ·60536	+ ·36613	− ·74902	+ ·30874	+ ·05542
		+ ·00112	− ·00357	− ·00186		+ ·00039	− ·00138	− ·00189		− ·35151	+ ·36695	− ·75643		+ ·38816	− ·87754	− ·00361	− 1·24120 [− 1·24117]
Totals		+ ·20722 $= \frac{1}{2}\delta_1 G$	− ·00526 $= \delta_2 G$	− ·00188 $= \frac{1}{2}\delta_3 G$		+ ·31491 $= \frac{1}{2}\delta_1 H$	+ ·00029 $= \delta_2 H$	− ·00188 $= \frac{1}{2}\delta_3 H$		− ·21613 $= \frac{1}{2}\delta_1 K$	+ ·37086 $= \delta_2 K$	− ·75647 $= \frac{1}{2}\delta_3 K$		− ·09612 $= \frac{1}{2}\delta_1 L$	− ·87730 $= \delta_2 L$	− ·00361 $= \frac{1}{2}\delta_3 L$	

The multiplications in any line may be checked by multiplying, *e.g.*, the sum of the coefficients of ω_2 in δG, δH, δK, δL into the sum of coefficients of ω_2, as given already, thus

$$+1·14882 - ·21691 + 4·90603 - ·92630 = +4·91164, \qquad +4·91164 \times -2·38174 = -11·69825.$$

The additions are to be checked by adding across, when they will give the same sums as the totals of the last column; the actual comparison is shown in [ ].

We conclude from this calculation that the aberrational terms in the emergent ray are

$$\begin{aligned}\delta b' &= b_0\,[+\text{·}20722\,(b_0^2+c_0^2)-\text{·}00526\,(b_0\beta_0+c_0\gamma_0)-\text{·}00188\,(\beta_0^2+\gamma_0^2)]\\ &\quad+\beta_0\,[+\text{·}31491\,(b_0^2+c_0^2)+\text{·}00029\,(b_0\beta_0+c_0\gamma_0)-\text{·}00188\,(\beta_0^2+\gamma_0^2)],\\ \delta\beta' &= b_0\,[-\text{·}21613\,(b_0^2+c_0^2)+\text{·}37086\,(b_0\beta_0+c_0\gamma_0)-\text{·}75647\,(\beta_0^2+\gamma_0^2)]\\ &\quad+\beta_0\,[-\text{·}09612\,(b_0^2+c_0^2)-\text{·}87730\,(b_0\beta_0+c_0\gamma_0)-\text{·}00361\,(\beta_0^2+\gamma_0^2)] \quad . \quad (27),\end{aligned}$$

with similar expressions for $\delta c'$, $\delta\gamma'$ if we replace b_0, β_0 outside the brackets by c_0, γ_0.

We may without loss of generality put $\gamma_0 = 0$, and in what follows this shall be done.

The twelve coefficients above are not independent. We have seen that they must satisfy seven relations, and we shall now verify that they do so.

We have, from p. 161,

$$\mathfrak{P} = k_0+nk_2+k_4+mk_6.$$

k	Term in $\mathfrak{P}$.
− ·41285	− ·41285
−1·58532	−1·03675
+1·14501	+1·14501
− ·54509	− ·33255
	− ·63714 = $\mathfrak{P}$

and thus

$\delta_2 G-\delta_1 H$ = − ·00526 − ·62982 = − ·63508		$\mathfrak{P}G$ = − ·63503
$\delta_3 G-\delta_2 H$ = − ·00376 − ·00029 = − ·00405		$\mathfrak{P}H$ = − ·00406
$\delta_2 K-\delta_1 L$ = + ·37086 + ·19224 = + ·56310		$\mathfrak{P}K$ = + ·56312
$\delta_3 K-\delta_2 L$ = −1·51294 + ·87730 = − ·63564		$\mathfrak{P}L$ = − ·63566

Next for the relations (20); we have, noting that N = 1,

s.	$+L\delta_s G$.	$-K\delta_s H$.	$-H\delta_s K$.	$+G\delta_s L$.	$=\delta_s N$.	
1	+ ·41347	+ ·55665	+ ·00275	− ·19160	+ ·78127	NK^2 = + ·78113
2	− ·00525	+ ·00026	− ·00236	− ·87440	− ·88175	NKL = − ·88176
3	− ·00376	− ·00332	+ ·00963	− ·00720	− ·00465	$N(L^2-1)$ = − ·00465

The comparisons seem to point to a small accumulated error in $\delta_1 H$.

If we transfer the origin for emergent rays to the focal plane, $x' = +1\text{·}12771$, we have finally

$$\begin{aligned}\delta b' &= b\,[-\text{·}03651\,d^2+\text{·}41296\,d\beta\cos\phi-\text{·}85496\beta^2]\\ &\qquad\qquad+\beta\,[+\text{·}20651\,d^2-\text{·}98905\,d\beta\cos\phi-\text{·}00595\beta^2]\\ \delta c' &= c\,[-\text{·}03651\,d^2+\text{·}41296\,d\beta\cos\phi-\text{·}85496\beta^2]\\ &\qquad\qquad+ \qquad\qquad * \qquad\qquad . \quad (28),\end{aligned}$$

where the subscript (0) is dropped, γ is taken as zero, and $b = d \cos \phi$, $c = d \sin \phi$, as on p. 165.

These give the aberrations of the lens at its principal focus.

The corresponding normal scheme is

$$\begin{aligned} b' &= \quad * \qquad +1{\cdot}131453\beta, \\ \beta' &= -{\cdot}883818b + {\cdot}997671\beta. \end{aligned} \qquad (29)$$

In order to fix ideas, compare (28) with the case of a parabolic reflector of the same focal length, given on p. 167, for which we get

$$\delta b' = b[\ *\ +{\cdot}44191b\beta +\ *\ ] + \beta[+{\cdot}22095d^2 - 1{\cdot}00000b\beta +\ *\ ];$$

we see that there is a close resemblance, except for the value of δ_3G, so that the two hardly differ in any sensible way, except in the curvatures of the fields. It will then cause some surprise that SEIDEL concluded that the Fraunhofer glass was free from coma which is so marked in the reflector. It was, in fact, a misapprehension, as the diagrams given by STEINHEIL and by FINSTERWALDER sufficiently demonstrate. SEIDEL'S argument presents an interesting feature.† He puts together the four components of his sum S (2),

$$\begin{array}{r} +\ 0{,}412 \\ -12{,}672 \\ +13{,}454 \\ -\ 1{,}662 \\ \hline \end{array}$$
$$S(2) = -\ 0{,}468$$

and draws his conclusion from the approximate balance, within one-thirtieth, of the large positive and negative members. It is evident, however, that this amounts to no more than saying that the two internal surfaces nearly annul one another. But the point I wish to make is that these numbers are in fact the same as those found on p. 178 above. If we transfer to the principal focus by adding $f'\delta$K to δG, we have for δ_2G

ω.	ψ.		δ_2G.	δ_2G $\times f$.
*	− ·36445	=	− ·36445	+ ·4125
−·12406	+11·32385		+11·19979	−12·6721
+·13396	−12·02515		−11·89119	+13·4543
−·00718	+ 1·47599		+ 1·46881	− 1·6619
			+ ·41296	+ 0·4672

The connection does not appear to be so close in the case of others of SEIDEL'S sums, but it is interesting to notice this common ground.

Let us now compare my calculations with those of STEINHEIL. First as to

† 'Ast. Nach.,' No. 1029, 326.

spherical aberration. Take rays parallel to the axis, that is, $\beta = 0$, and consider the focus where they unite for impact upon the original plane at fractions 0, 1/3, 2/3, and 1 of the semiaperture, *i.e.*, for

$$d = 0{\cdot}00000, \qquad 0{\cdot}01170, \qquad 0{\cdot}02340, \qquad 0{\cdot}03510.$$

For any of these the distance of the point from the last surface (6) is

$$-(\mathrm{G}+\tfrac{1}{2}\delta_1\mathrm{G}d^2)/(\mathrm{K}+\tfrac{1}{2}\delta_1\mathrm{K}d^2)$$

or

$$-\mathrm{G}/\mathrm{K}\,[1+\tfrac{1}{2}d^2\,(\delta_1\mathrm{G}/\mathrm{G}-\delta_1\mathrm{K}/\mathrm{K})], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and

$$\mathrm{G} = +{\cdot}996692, \qquad \mathrm{K} = -{\cdot}883818, \qquad \tfrac{1}{2}\delta_1\mathrm{G} = +{\cdot}20722, \qquad \tfrac{1}{2}\delta_1\mathrm{K} = -{\cdot}21613.$$

$$\therefore \tfrac{1}{2}[\delta_1\mathrm{G}/\mathrm{G}-\delta_1\mathrm{K}/\mathrm{K}] = +{\cdot}20791-{\cdot}24454 = -{\cdot}03663.$$

Hence the rays meet at the following points along the axis:—

		STEINHEIL, p. 417.	
Axial	$1127^l{\cdot}712$	$1127^l{\cdot}712$	
1/3 semiaperture . .	·706	·706	
2/3 ,, . .	·689	·687	
1 ,, . .	·662	·659. . . .	(31)

In these and the following comparisons the unit of length has been brought back to 1 line by multiplying by 1000, to preserve STEINHEIL'S numbers unchanged.

In consequence of this residue of spherical aberration the best setting for focus at the middle of the field is not the axial focus but a point within it. STEINHEIL takes this point at 1127·670, following presumably the theory of BESSEL, which gives a position for the greatest apparent concentration of light that is slightly within the least circle of aberration (1127·672).*

Adopting the corresponding point, which allows for the slightly smaller aberration shown by my numbers, and multiplying by 10^{-3} to bring the units into agreement with formula (28), we see, in accordance with p. 165, we must include with $\delta b'$ of p. 179 the term

$$+\mathrm{K}\delta f'\,.\,b = -{\cdot}8838\times-{\cdot}0000400\times b = +{\cdot}000035 35b,$$

with a corresponding term for $\delta c'$ in terms of c.

The diameters of the image-disc in the focal plane and at this setting are respectively the corresponding extreme values of $\delta b'$, doubled, or

	STEINHEIL.
$0^l{\cdot}00316$	—
$0^l{\cdot}00067$	$0^l{\cdot}00071.$

* BESSEL, *loc. cit.*, p. 104.

We take next the oblique rays in a plane through the axis, that is, we take $c = 0$. The rays considered are taken at an angle of 48′ with the axis, following BESSEL and STEINHEIL; hence $\beta = \tan 48' = {}^{\cdot}01396353$, and we take in succession $b = d = +{}^{\cdot}03510$, $+{}^{\cdot}02340$, $+{}^{\cdot}01170$, $-{}^{\cdot}01170$, $-{}^{\cdot}02340$, $-{}^{\cdot}03510$. The final numbers given below have been multiplied by 1000 in order to compare with STEINHEIL.

The central ray meets the chosen plane at a distance from the axis

$$(\mathrm{H}+\mathrm{L}\delta f')\,\beta = (+1{}^{\cdot}131453-{}^{\cdot}000040)\times\beta = {}^{\cdot}01579852.$$

Giving as before the calculations in full, the formulæ (28), supplemented by the term $\mathrm{K}\delta f'\,.\,b$, give the following:—

$\delta b'$—

b.	$+\mathrm{K}\delta f'$.	$+\frac{1}{2}\delta_1\mathrm{G}d^2$.	$+\delta_2\mathrm{G}b\beta$.	$+\frac{1}{2}\delta_3\mathrm{G}\beta^2$.	Coeff. b.	$+\frac{1}{2}\delta_1\mathrm{H}d^2$.	$+\delta_2\mathrm{H}b\beta$.	$+\frac{1}{2}\delta_3\mathrm{H}\beta^2$.	Coeff. β.
+ ·03510	+353,5	− 449,8	+2023,9	− 1667,2	+ 260,4	+2544,2	− 4847,3	− 11,6	− 2314,7
·02340	*ibid.*	− 199,9	+1349,3	*ibid.*	− 164,3	+1130,8	− 3231,5	*ibid.*	− 2112,3
+ ·01170	*ibid.*	− 50,0	+ 674,6	*ibid.*	− 689,1	+ 282,7	− 1615,8	*ibid.*	− 1344,7
− ·01170	*ibid.*	− 50,0	− 674,6	*ibid.*	− 2038,3	+ 282,7	+ 1615,8	*ibid.*	+ 1886,9
·02340	*ibid.*	− 199,9	− 1349,3	*ibid.*	− 2862,9	+1130,8	+3231,5	*ibid.*	+4350,7
− ·03510	*ibid.*	− 449,8	− 2023,9	*ibid.*	− 3787,4	+2544,2	+4847,3	*ibid.*	+7379,9

The comma is placed between the 7th and 8th decimals.

Hence we have the following, with unit 1 line:—

b.	$\delta b'$.	$b'+\delta b'$.	STEINHEIL, p. 419.
+ 35^l·1	+ 91 − 323 = − ·00232	15^l·79628	15^l·79622
23 ·4	− 38 − 295 − 333	·79519	·79528
+ 11 ·7	− 81 − 188 − 269	·79583	·79587
0 ·0	0 − 2 − 2	·79850	·79852
− 11 ·7	+ 238 + 263 + 501	·80353	·80349
23 ·4	+ 669 + 608 + 1277	·81129	·81130
− 35 ·1	+ 1329 + 1031 + 2360	·82216	·82212

. (32)

There is a slight discrepancy for $b = +23{}^{\cdot}4$.

STEINHEIL now considers the rays which do not meet the axis. Fig. 6 is taken from his memoir and shows the object-glass on reduced linear scale. He divides the object-glass into three rings, and computes all the rays which impinge upon it at an angle of 48′ with the axis, at the points indicated in the figure. The rays 2, 10, 18, 1, 22, 14, 6 are those just given; of the remainder, those upon the left may be written down from symmetry from those upon the right, so that he computes in all nine independent rays which do not meet the axis.

We derive these as follows. We have throughout β unchanged :—

For the rays (4), (12), (20), $b = 0$, $c = d$;
,, ,, (3), (11), (19), $b = c = d \sin 45^\circ = d \times \cdot 70711$,
,, ,, (5), (13), (21), $-b = c = d \sin 45^\circ = d \times \cdot 70711$.

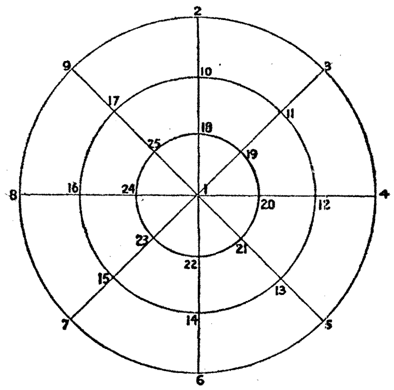

Fig. 6.

Hence the additional calculations required run as follows.

In the calculation of the coefficients replace the columns $\delta_2 \mathrm{G} b \beta$, $d_2 \mathrm{H} b \beta$ by the following :—

Ray.	$\delta_2 \mathrm{G} b \beta$.	$\delta_2 \mathrm{H} b \beta$.
3	+ 1431,1	− 3427,6
11	+ 954,1	− 2285,0
19	+ 477,0	− 1142,5
21	− 477,0	+ 1142,5
13	− 954,1	+ 2285,0
5	− 1431,1	+ 3427,6

In the rays (4), (12), (20) replace these by zero.

Therefore

Ray.	Coefficient b or c.	Coefficient β.	$= \delta b'$.	$\delta c'$.
3	− 332,4	− 895,0	− 82 − 125 = − l·00207	− l·00082
11	− 559,5	− 1165,8	− 93 − 163 = − ·00256	− ·00093
19	− 886,7	− 871,4	− 73 − 122 = − ·00195	− ·00073
21	− 1840,7	+ 1413,6	+ 152 + 197 = + ·00349	− ·00152
13	− 2467,7	+ 3404,2	+ 408 + 475 = + ·00883	− ·00408
5	− 3194,6	+ 5960,2	+ 793 + 832 = + ·01625	− ·00793
4	− 1763,5	+ 2532,6	+ * + 354 = + ·00354	− ·00618
12	− 1513,6	+ 1119,2	+ * + 156 = + ·00156	− ·00354
20	− 1363,7	+ 271,1	+ * + 038 = + ·00038	− ·00160

STEINHEIL.

$\delta b'$.	$\delta c'$.
− l·00204	− l·00080
− ·00251	− ·00092
− ·00191	− ·00070
+ ·00351	− ·00151
+ ·00879	− ·00410
+ ·01605	− ·00758
+ ·00357	− ·00627
+ ·00160	− ·00353
+ ·00041	− ·00162

(33)

There is a slight discrepancy in the ray (5).

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In considering what discrepancies may be expected, we have to recall that the method developed in the preceding pages omits terms of the fifth order, which may amount to, say,

$$coeff. \times d \times \cdot 000001.$$

Taking $d = 35^{l \cdot}1$, for unity as coefficient, we should have an error of 4 units in the last place retained above. We have no means of saying what the coefficient may be, but it is clear that it may affect the last digit. Yet I believe that these calculations are not only very much easier, but also more correct than the trigonometrical ones, for though the formulæ for the latter are exact, the number of operations they require is very large. Thus, for each ray which meets the axis, there are fully 50 operations of which at least one-half consist in taking out a logarithm or an antilogarithm with seven decimal places; for each ray which does not meet the axis the work is rather more than four times as great.

STEINHEIL has calculated seven of the former rays and nine of the latter.

The controls that exist are of the most meagre description and give little help in locating an error. But, even if the whole is done in strictest accordance with the tables, at any step an error may be introduced which falls only short of half a unit in the last place. Thus, in the rays which do not meet the axis, an irremovable accumulated error of 10 or more units could cause no surprise, and for this reason the trigonometrical method loses any advantage over the formulæ given above which it might claim from resting upon exact formulæ. The differences under discussion are, however, minimal, since 550 units in the last decimal place only amount to 1 second of arc.

But pursuing the question a little further I believe, in spite of the evident care with which the whole of STEINHEIL's calculations have been carried through, that the comparison above shows that a small error has crept in in respect to ray (5).

If we take the general agreement as showing that the trigonometrical calculation does in fact bring in no terms of the aberrations beyond the 3rd order, we can readily analyse STEINHEIL's numbers in more than one way so as to derive the coefficients δ_1G, ... from them.

Take the formulæ (24); on the outer ring $\phi = 0, 45°, 90°, 135°, 180°$, correspond respectively to the rays 2, 3, 4, 5, 6; thus we have

$$d^2\beta\delta_2 G = \delta c'_3 - \delta c'_5 = \tfrac{1}{2}(\delta b'_2 + \delta b'_6) - \delta b'_4$$

$$K d\delta f + \tfrac{1}{2}d^3\delta_1 G + \tfrac{1}{2}d\beta^2\delta_3 G = \qquad = \frac{1}{\sqrt{2}}(\delta c'_3 + \delta c'_5) = \delta c'_4$$

$$K d\delta f + \tfrac{1}{2}d^3\delta_1 G + \tfrac{1}{2}d\beta^2(\delta_3 G + 2\delta_2 H) = \qquad = \frac{1}{\sqrt{2}}(\delta b'_3 - \delta b'_5) = \tfrac{1}{2}(\delta b'_2 - \delta b'_6)$$

$$d^2\beta(\delta_1 H + \delta_2 G) + \beta^3\delta_3 H = \delta b'_3 + \delta b'_5 = \tfrac{1}{2}(\delta b'_2 + \delta b'_6) + \delta b'_4$$

$$\beta^3\delta_3 H = \delta b'_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

From these we get at once

$$d^2\beta\delta_2 G, \quad \beta^3\delta_3 H, \quad \text{and} \quad d\beta^2\delta_2 H,$$

and thence

$$K d\delta f + \tfrac{1}{2}d^3\delta_1 G + \tfrac{1}{2}d\beta^2\delta_3 G \quad \text{and} \quad d^2\beta\delta_1 H.$$

Also for the case $\beta = 0$,

$$\delta b'_2 = K d\delta f' + \tfrac{1}{2}d^3\delta_1 G\,;$$

thus we get $d\beta^2\delta_3 G$, and when the adopted value of $\delta f'$ is used, $d^3\delta_1 G$ also, which completes the solution.

We see that we can use the rays (A)—3, 5 exclusively, or (B)—2, 4, 6 exclusively. Making separate determinations by these roads,

	(A.)	(B.)
$d^2\beta\delta_2 G$	$+\cdot 00678$	$+\cdot 00709$
$K d\delta f' + \frac{1}{2}d^3\delta_1 G + \frac{1}{2}d\beta^2\delta_3 G$	$-\cdot 00593$	$-\cdot 00627$
$K d\delta f' + \frac{1}{2}d^3\delta_1 G + \frac{1}{2}d\beta^2(\delta_3 G + 2\delta_2 H)$ . . .	$-\cdot 01279$	$-\cdot 01295$
$d^2\beta(\delta_1 H + \delta_2 G) + \beta^3\delta_3 H$	$+\cdot 01401$	$+\cdot 01422$

$\beta^3\delta_3 H$	$\cdot 00000$
$2K d\delta f' + d^3\delta_1 G$	$-\cdot 00071$
$2K d\delta f'$	$+\cdot 00260$

Hence

	(A.)	(B.)		(A.)	(B.)	p. 179.
$d^3\delta_1 G$ . .	$-\cdot 00331$		$\delta_1 G$ . .	$-\cdot 0766$		$-\ \cdot 0730$
$d^2\beta\delta_2 G$ . .	$+\cdot 00678$	$+\cdot 00709$	$\delta_2 G$ . .	$+\ \cdot 3941$	$+\ \cdot 4121$	$+\ \cdot 4130$
$d\beta^2\delta_3 G$ . .	$-\cdot 01115$	$-\cdot 01183$	$\delta_3 G$ . .	$-1\cdot 6292$	$-1\cdot 7285$	$-1\cdot 7099$
$d^2\beta\delta_1 H$ . .	$+\cdot 00723$	$+\cdot 00713$	$\delta_1 H$ . .	$+\ \cdot 4202$	$+\ \cdot 4144$	$+\ \cdot 4130$
$d\beta^2\delta_2 H$ . .	$-\cdot 00686$	$-\cdot 00668$	$\delta_2 H$ . .	$-1\cdot 0023$	$-\ \cdot 9760$	$-\ \cdot 9891$
$\beta^3\delta_3 H$ . .	$\cdot 00000$		$\delta_3 H$ . .	$\cdot 0000$		$-\ \cdot 0060$

There is no doubt, from the checks on p. 179, that the numbers put in the last column for comparison are correct to the last digit, and we see that the numbers (A) which rest upon the ray (5) are decidedly less consistent with them than the numbers (B) which do not.